LECTURES IN RELATIVITY AND GRAVITATION

A MODERN LOOK

Anatoly Logunov

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PREFACE

In the history of physics major landmarks are associated with contributions of the greatest natural philosophers, who on the basis of all previous results and novel revolutionary ideas and methods made another significant breakthrough in human cognition of Nature.

Great accomplishments of the human intellect always call for thorough and creative examination, for these achievements enable one to trace the conception and evolution of epoch-making ideas. Studying of classics is a wonderful school, if only it involves thoughtful probing into the essence of a problem, rather than blind faith in authority. The latter always leads to dogmatism. A detailed creative analysis of classical works may at times reveal an incompleteness of ideas, and even profound mistakes, and this in turn may give rise to a new direction in research. But science history also knows of cases when a researcher, having failed to grasp the import of a work, jumped to the conclusion that the author had misunderstood the problem at hand, and even made claims to the credit of the discovery. But if somebody could not understand the work, the author is not to blame. After all, one's own misunderstanding should not be a criterion for appraizing a work, especially a classic one. For an investigator misunderstanding is always of value, because it may provide an impetus for an in-depth analysis of a problem. Profound misunderstanding may produce something new and significant, and nearly always something interesting, which may enrich the researchers with added knowledge or a new approach. However, this always requires serious effort and detailed physical analysis.

Of course, people differ in their creative perception, and so understanding somebody is also some kind of creative process, which is not always an easy task. There is only one way – seek to penetrate deeper into the essence of a work. To gain an understanding of the classic work, the reader may need to learn some additional things. If he still fails, which is quite probable, this may then be an indication of his specific perception or, alas, of the inadequacy of scientific level. But either reason is purely personal and has nothing to do with the cognition of scientific truth.

Unfortunately, the foregoing is what occurs quite often in physics; therefore, not to find yourself lost you are strongly recommended to cover new ground painstakingly, and if necessary several times.

This approach to mastering the material will mould you into an independent and free thinker learning on solid knowledge. The example of Relativity will show you how formal and shallow assimilation of some classic works has led to grave delusions. One object of my lectures is to free you of them.

The book includes some lectures on the basics of relativity theory, or as it is generally called the special theory of relativity. Relativity theory has been created by the outstanding scientists Lorentz [1], Poincaré [2-3], Einstein [4], and Minkowski [6]. These giants, I think, have virtually completed the theory, and what came after was interpretation, once correct, once not, but nearly always superficial.

In modern textbooks and monographs relativity theory is sometimes presented in a trivial and limited manner. Not infrequently the authors fail to bring out the principal and get tangled in secondary problems. One may gather, reading those texts, that the theory is just a collection of recipes, which are sometimes hard to grasp for their limitedness. That is why I begin with the relativity postulate (Section 1.3), which cannot be proved and simply follows from analysis of experimental results. It should be adequately absorbed so that it might then be applied to specific phenomena.

What is covered by the lectures could have been accomplished long ago, after Minkowski's work, and he might have expounded all this himself, had he not died so untimely. However dogmatism and faith – two things that have at all times been foreign to science although constantly by plagued it – have had their effect. So, nearly to these days they have drastically reduced the level of understanding and, as a consequence, have narrowed the domain of application of relativity theory. Only after assimilating the basics of Minkowski's work, and what is presented in sufficient detail in the lectures, one can arrive at the general formulation that the theory of relativity is the discovery of a unified pseudo-Euclidean geometry of space-time for electromagnetic phenomena and its generalization, as a hypothesis, to all forms of matter.

It is shown in the lectures that clock synchronization, a topic that is generally attached all too much importance in texts on relativity, is a partial question. As regards the postulate on the constancy of the velocity of light, even if given a correct formulation, as in these lectures, it plays a limited role, since it only makes sense for inertial reference frames. Outside these frames there is no use for it.

On the other hand, the views of the pseudo-Euclidean geometry of a unified space-time are more general and fundamental. They allow us to put into a similar perspective both inertial and accelerated frames of reference and to formulate the generalized relativity principle. The extension of the scope of special relativity theory is not only of fundamental but also of applied importance, since we can now look at phenomena under some extreme conditions.

The book has grown out of a course read at the physics department of Moscow University in 1983-84. Hence some inevitable redundancies, for which the author offers his apologies. The last chapter covers some new results in the relativistic theory of gravitation.

A. LOGUNOV

Chapter 1

SPACE AND TIME

1.1. Space and Time in Newton's Mechanics

Relativity principle is one of the most ancient and fundamental principles of modern physics. Its beginnings data back to the days of natural philosophy. The emergence and evolution of the principle is closely linked with the development of our ideas of space and time, since space and time are the scene on which all physical processes are played out. Therefore, as our physical knowledge developed, so did our views on space and time. Our outline of the development of man's concepts of space and time begins with Newton's mechanics.

Mechanics, as a science of the notion of bodies, started to grow vehemently in the middle of the 17th century. The mechanics of the day was an experimental science concerned with establishing empirical relationships between kinematic and dynamic characteristics of travelling bodies and forces on them. Having analyzed multiple experimental data Newton formulated his three celebrated laws of dynamics and the law of gravitation. This made it possible to handle a wide variety of problems on the motion of bodies.

Newtonian mechanics also constituted a milestone in the evolution of our concepts of space and time. On the one hand, it used the Euclidean geometry of three-dimensional space as a consequence of its reliance on the rules of vector composition and the determination of the distance between two points. On the other, experimental tests of the main principles of Newton's mechanics and comparisons of its predictions with experiment showed with great accuracy that three-dimensional space is indeed Euclidean.

The properties of space and time in Newton's mechanics can also be established from an examination of a group of transformations that leave the equations of motion form-invariant. Write

Newton's equations, say, for a system of two particles

$$
\begin{aligned}
M_1 \frac{d^2\mathbf{r}_1}{dt^2} &= F(|\mathbf{r}_2 - \mathbf{r}_1|)\,\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \\
M_2 \frac{d^2\mathbf{r}_2}{dt^2} &= -F(|\mathbf{r}_2 - \mathbf{r}_1|)\,\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|},
\end{aligned}
\tag{1.1.1}
$$

where $\mathbf{r}_1$ is the position vector of a first particle, and $\mathbf{r}_2$ is the position vector of the second one; M_1 and M_2 are the masses of the first and second particles respectively.

The function $F(\mathbf{r}_2-\mathbf{r}_1)$ characterizes the nature of the forces between the bodies. Newtonian mechanics deals mostly with two types of forces – gravity and elastic. For gravity forces we have

$$
F(|\mathbf{r}_1 - \mathbf{r}_2|) = -\gamma \frac{M_1 M_2}{|\mathbf{r}_2 - \mathbf{r}_1|^2},
$$

for the elastic force

$$
F(|\mathbf{r}_2 - \mathbf{r}_1|) = -k\,|\mathbf{r}_2 - \mathbf{r}_1|,
$$

where k is the elastic constant and γ is the gravitational constant.

Expressions (1.1.1) suggest that they are form-invariant under transformations of the origin of coordinates, when time remains unchanged, and the coordinates are shifted by a constant vector

$$
\mathbf{r}' = \mathbf{r} + \mathbf{b}, \quad t' = t. \tag{1.1.2}
$$

Indeed, under this transformation the position vector of the bodies will be

$$
\begin{aligned}
\mathbf{r}_1' &= \mathbf{r}_1 + \mathbf{b}, \\
\mathbf{r}_2' &= \mathbf{r}_2 + \mathbf{b}.
\end{aligned}
$$

It follows that the difference of these position vectors will not change under the transformation (1.1.2): $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1' - \mathbf{r}_2'$. Moreover, by the first of (1.1.2) we have $d/dt = d/dt'$. Therefore, equations (1.1.1) under the transformation (1.1.2) become

$$
\begin{aligned}
M_1 \frac{d^2\mathbf{r}_1'}{dt'^2} &= F(|\mathbf{r}_2' - \mathbf{r}_1'|)\,\frac{\mathbf{r}_1' - \mathbf{r}_2'}{|\mathbf{r}_1' - \mathbf{r}_2'|}, \\
M_2 \frac{d^2\mathbf{r}_2'}{dt'^2} &= -F(|\mathbf{r}_2' - \mathbf{r}_1'|)\,\frac{\mathbf{r}_1' - \mathbf{r}_2'}{|\mathbf{r}_1' - \mathbf{r}_2'|}.
\end{aligned}
\tag{1.1.3}
$$

Comparing (1.1.1) and (1.1.3) we see that they have the same functional dependence on the coordinates, and so, to use the language of modern physics, are form-invariant under transformations (1.1.2). Note also that the transformations (1.1.2) keep the magnitude of the relative velocity unaltered:

$$\frac{d}{dt}(\mathbf{r}_1 - \mathbf{r}_2) = \mathbf{v} = \frac{d}{dt'}(\mathbf{r}_1' - \mathbf{r}_2') = \mathbf{v}'.$$

There are two treatments of the physical meaning of transformation (1.1.2): (a) description of a mechanical phenomenon is independent of where we have placed the origin of coordinates; (b) if we translate the mechanical phenomenon to another point in space, separated from the first one by vector $\mathbf{b}$, it will proceed there in the same manner. This implies that there are no distinguished points in space, and so it is homogeneous.

Second, under a shift in time

$$t' = t + a; \quad \mathbf{r}' = \mathbf{r} \tag{1.1.4}$$

equations (1.1.1) also remain form-invariant, since under the transfor-

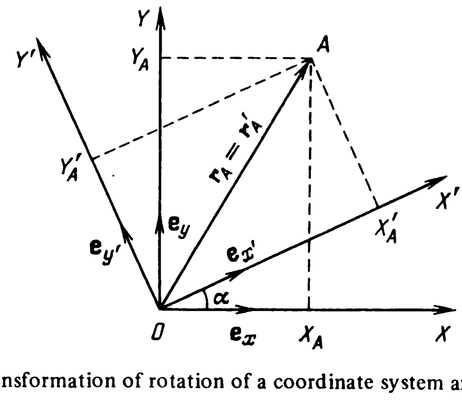

Fig 1. Transformation of rotation of a coordinate system around an axis

mation (1.1.4) differentials with respect to time coincide:

$$\frac{d}{dt} = \frac{d}{dt'}.$$

The magnitude of the relative velocity for the two bodies is not changed by the transformation as well. Transformation (1.1.4) describes

the translation of the zero time for all clocks by the same time period, without changing the space coordinates. Therefore, the form-invariance of the equations of mechanics under this transformation suggests that there are no distinguished times and all the points on the time axis are physically equivalent, with the result that a mechanical process with the same initial conditions will occur in the same manner, regardless of when we begin it. In other words, two mechanical processes under the same initial conditions will proceed in the same manner, whatever the choice of the initial time for each of them.

Further, from this mechanics we infer that equations (1.1.1) are invariant under rotations of the frame of reference through arbitrary angles. In the special case of rotating the frame about the z-axis by an angle α (see Fig. 1) we have

$$\begin{aligned} X'_A &= X_A \cos\alpha + Y_A \sin\alpha, \\ Y'_A &= Y_A \cos\alpha - X_A \sin\alpha, \\ \mathbf{e}_{x'} &= \mathbf{e}_x \cos\alpha + \mathbf{e}_y \sin\alpha, \\ \mathbf{e}_{y'} &= \mathbf{e}_y \cos\alpha - \mathbf{e}_x \sin\alpha, \qquad t = t', \end{aligned} \tag{1.1.5}$$

therefore

$$\mathbf{r}'_A = X'_A\,\mathbf{e}_{x'} + Y'_A\,\mathbf{e}_{y'} = X_A\,\mathbf{e}_x + Y_A\mathbf{e}_y = \mathbf{r}_A,$$

and so the transformation makes equations (1.1.2) unchanged. Put another way, rotating the original frame of reference by a constant angle does not change the course of the mechanical process at hand. This indicates that space is isotropic, i.e., it has no distinguished directions.

The equations of Newtonian mechanics thus enable us to make some conclusions regarding the properties of space and time: space is homogeneous and isotropic and time is homogeneous. If space did not possess these properties, Newtonian mechanics, which is based on equations (1.1.1) might be wrong to some degree of accuracy.

There is, however, a fourth group of transformations under which equations (1.1.1) are form-invariant, which in the mechanics of Newton has always stood somewhat apart, namely Galilean transformations. They describe transition from one frame of reference to another that moves uniformly along a straight line:

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t, \quad t' = t. \tag{1.1.6}$$

We then have

$$\frac{d}{dt} = \frac{d}{dt'}, \quad \mathbf{r}'_1 - \mathbf{r}'_2 = \mathbf{r}_1 - \mathbf{r}_2. \tag{1.1.7}$$

Substituting (1.1.7) into equations (1.1.1) we can easily see that transformation (1.1.6) keeps the equations of mechanics form-invariant. But whereas the transformations (1.1.2), (1.1.4) and (1.1.5) were concerned with either three-dimensional space or time, this transformation was regarded as additional following from the equation of Newtonian mechanics. This transformation was never completely perceived. True, it immediately leads to a confirmation of the relativity principle in mechanics, since all mechanical processes considered in one inertial frame of reference are virtually the same as in another frame that moves uniformly in a straight line. When you are investigating mechanical phenomena you cannot say whether you are at rest or travel uniformly, since transformations (1.1.6) leave the equations of Newton's mechanics unaltered in form, and the equations do not include the velocity **V** of motion of one frame relative to the other. And so transformations (1.1.1) in Newton's mechanics formed the foundation of Galileo's relativity principle.

The two groups of transformations just considered, namely the one reflecting the property that time is homogeneous and that space is homogeneous and isotropic and Galileo's group, have existed in their own right. For a long time, until the works of Poincaré and Minkowski, the true relation between them was not established.

Also, it follows directly from Newton's mechanics that distance in a three-dimensional space, i.e., in a coordinate space, is an invariant for all inertial frames of reference.

Indeed, the distance between points A and B in the "unprimed" (stationary) frame of reference is

$$l^2 = (X_B - X_A)^2 + (Y_B - Y_A)^2 + (Z_B - Z_A)^2 .$$

In the "primed" (moving) frame of reference we have

$$l'^2 = (X'_B - X'_A)^2 + (Y'_B - Y'_A)^2 + (Z'_B - Z'_A)^2 .$$

Substituting the relations

$$X'_B = X_B - V_x t, \quad Y'_B = Y_B - V_y t, \quad Z'_B = Z_B - V_z t$$

and similar relations between primed and unprimed coordinates of point A, we will get

$$l = l'. \tag{1.1.8}$$

Consequently, the concept of length is absolute, i.e., it is independent of the frame of reference. As regards time, it enters into the relations as a kind of parameter, which is also independent of frame of reference and which is the same in different frames.

Newtonian mechanics thus introduces the absolute notion of distance between points in three-dimensional space and absolute time. We have seen earlier that the equations of Newton's mechanics are invariant against a group of transformations of three-dimensional space, which was an indication of space being Euclidean. We will now establish this fact from the Hamilton–Jacobi formalism of mechanical equations. It is well known in mechanics that the Hamilton function can be written in terms of the Lagrange function as

$$H = p_k \dot{q}_k - L,$$

where $p_k = \partial L / \partial \dot{q}_k$ is the generalized momentum.

The Hamilton equations in mechanics are symmetrical in form:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = \frac{\partial H}{\partial q_k}. \tag{1.1.9}$$

Before going over to the Hamilton–Jacobi form of mechanical equation, we will remind you of some facts from the theory of the first order partial differential equations. If an equation has the form

$$F(q_1, \ldots, q_n; p_1, \ldots, p_n) = 0, \quad p_i = \frac{\partial S}{\partial q_i}, \tag{1.1.10}$$

the equations of its characteristics are a system of ordinary differential equations

$$\frac{dq_1}{\dfrac{\partial F}{\partial p_1}} = \ldots = \frac{dq_n}{\dfrac{\partial F}{\partial p_n}} = - \frac{dp_1}{\dfrac{\partial F}{\partial q_1}} = \ldots = \frac{dp_n}{\dfrac{\partial F}{\partial q_n}}. \tag{1.1.11}$$

To derive the Hamilton–Jacobi equation we will write the Hamilton system (1.1.9) in the form

$$\frac{dt}{1} = \frac{dq_1}{\dfrac{\partial H}{\partial p_1}} = \ldots = \frac{dq_n}{\dfrac{\partial H}{\partial p_n}} = - \frac{dp_1}{\dfrac{\partial H}{\partial q_1}} = \ldots = - \frac{dp_n}{\dfrac{\partial H}{\partial q_n}}. \tag{1.1.12}$$

Using the above relations we can replace this system by the partial differential equation of the first order

$$\frac{\partial S}{\partial t} + H\left(t, q_1, \ldots, q_n; \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}\right) = 0. \tag{1.1.13}$$

This is the Hamilton–Jacobi equation.

For one particle travelling in the Cartesian system of coordinates we have

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2).$$

Hence

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] = 0. \tag{1.1.14}$$

If we change to curvilinear coordinates, we will obtain

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\gamma^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} = 0. \tag{1.1.15}$$

Here γ^{ik} is a metric tensor.

The distance between two neighboring points in space is expressed through this tensor in the following manner (the repeated indices mean summation):

$$dl^2 = \gamma^{\alpha\beta} dx_\alpha \, dx_\beta = \gamma_{\alpha\beta} dx^\alpha dx^\beta. \tag{1.1.16}$$

Because in a Cartesian system of coordinates the metric tensor is a diagonal one, i.e.,

$$\gamma_{11} = \gamma_{22} = \gamma_{33} = 1, \quad \gamma_{i,k} = 0, \quad \text{if} \neq k \tag{1.1.17}$$

then the three-dimensional space is Euclidean.

In summary, if for some form of matter we have laws of its motion in terms of differential equations, then these equations have incorporated into them notions of the structure of space and time. Although this statement now seems quite natural and general, the path to it was not all roses. It is remarkable that the path turned out to be harded for physicists and easier for mathematicians concerned with natural sciences. This can be accounted for by the fact that by the time mathematicians had fully developed the theory of groups and invariants and had been concerned with the group of motion in space, whereas for the physicists of the day the notions of invariance and group had been yet foreign.

As we have seen, the equations of mechanics are form-invariant with respect to Galilean transformations, and so for all mechanical processes the relativity principle is valid. Poincaré formulated the principle for all physical phenomena in the following manner [5]: "The relativity principle, which states that the laws of physical phenomena

must be the same for a stationary observer and for an observer who undergoes uniform translational motion, so that we do not have, and cannot have, a way of determining whether or not we are in motion."

Since Maxwell's equations, which describe electromagnetic phenomena, change under Galilean transformations, it was concluded that Maxwell's equations do not satisfy the relativity principle. Attempts have been made to modify the equations so that they would remain unchanged under Galilean transformations. Such new equations were derived by Herz. The equations acquired new terms, which suggested that there must be some new electromagnetic processes in nature. Experiment did not confirm this, however. All doubts in the validity of Maxwell's equations gradually disappeared.

1.2. Maxwell–Lorentz Electrodynamics and Minkowski Unified Space-Time

Generalizing experimental evidence and profound insights of Faraday's, Maxwell combined magnetic, electric and optic phenomena and derived his famous equations, which for an electron moving in a stationary frame of reference were written by Lorentz as follows:

$$\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\rho\mathbf{v} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \quad \operatorname{div}\mathbf{H} = 0, \tag{1.2.1}$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \qquad \operatorname{div}\mathbf{E} = 4\pi\rho,$$

$$\mathbf{f} = \rho\mathbf{E} + \frac{\rho}{c}[\mathbf{vH}],$$

where $\mathbf{E}$ is electric field and $\mathbf{H}$ is the magnetic field, ρ is the volume density of an electron, $\mathbf{f}$ is the electrodynamic force exerted by the field on a volume element of the electron (Lorentz's force).

These equations are known as the Maxwell–Lorentz equations. Galilean transformations do not leave them unchanged. This seems to suggest that they do not satisfy the relativity principle. But this conclusion is wrong. Galilean transformations and the relativity principle are two different things. Galilean transformations have been derived from Newton's equations as transformations that leave them unaltered in changing from a fixed reference frame to a frame that moves with a constant velocity, and if Newton's equations are not always obeyed in nature and so will be replaced by other laws of mechanics, Galilean transformations need not provide for the invariance of new laws of mechanics. The relativity principle has a more fundamental

nature. According to it, even if Newton's equations are replaced by other mechanical equations, no mechanical phenomena can show whether you are at rest or in uniform motion. The impossibility of existence of absolute motion is a general law of nature. This means that electromagnetic phenomena too can never establish absolute motion. Experimental data indicated that the Maxwell–Lorentz equations adequately describe electromagnetic and optical phenomena.

Consider two frames of reference: we will think of one as stationary (K)and of another (K) as moving relative to the first one uniformly and rectilinearly along the X-axis with a velocity ϵ. If the relativity principle holds for electromagnetic phenomena as well, the Maxwell–Lorentz equations must have the same form both in the stationary frame K and in the travelling frame K', because it is only under these conditions that electromagnetic processes will occur identically in K and K' provided the initial and boundary conditions are appropriate.

The problem thus is to show that the Maxwell–Lorentz equations are invariant in changing from K to K'. The problem was first posed and solved by Poincaré. He showed, following Lorentz, that if coordinates and time are transformed in the following manner (Lorentz transformations):

$$X' = \gamma(X + \epsilon T), \quad Y' = Y, \quad Z' = Z, \quad T' = \gamma(T + \epsilon X)$$

(the velocity of light is here taken to be unity) and if by the very same Lorentz transformation we change the vector and scalar potentials (A, φ)

$$A'_x = \gamma(A_x + \epsilon\varphi), \ A'_y = A_y, \ A'_z = A_z, \ \varphi' = \gamma(\varphi + \epsilon A_x),$$

the current and charge density

$$\rho' v'_x = \gamma(\rho v_x + \epsilon\rho), \ \rho' v'_y = \rho v_y,$$
$$\rho' v'_z = \rho v_z, \ \rho' = \gamma(\rho + \epsilon\rho v_x),$$

and also the Lorentz force and work in a unit time

$$f'_x = \gamma(f_x + \epsilon \mathbf{f}\mathbf{v}), \ f'_y = f_y, \ f'_z = f_z,$$
$$\mathbf{f}'\mathbf{v}' = \gamma(\mathbf{f}\mathbf{v} + \epsilon f_x),$$

then in K' we will have again the Maxwell–Lorentz equations, and so electromagnetic phenomena occur in K and K' in the same manner under appropriate initial and boundary conditions. Poincaré observes [3]: "Two frames – one stationary, the other moving translationally – are thus exact images of each other."

The relativity principle for electromagnetic phenomena thus follows from the Maxwell–Lorentz equations as a rigorous mathemat-

ical truth. It is worth noting that with the Lorentz transformations for the origin of the new coordinates ($X' = Y' = Z' = 0$) we have

$$X = -\epsilon T.$$

It follows that the origin of K' moves along the X-axis of the frame K with velocity ϵ. The Lorentz transformations relate the coordinates and time (X', Y', Z', T') of the frame K' to the coordinates and time (X, Y, Z, T) of the frame K. Poincaré established that the Lorentz transformations form a group together with all the other space transformations. It should be stressed that one trivial consequence of the fact that Maxwell's equations are invariant under the Lorentz transformations is that the relativity principle holds for electromagnetic phenomena. If the laws of nature are invariant with respect to the Lorenz transformations, it means that the relativity principle holds for all natural phenomena. The opinion, often to be encountered in the literature, that relativity theory should forsake Galilean transformations is ungrounded. These transformations, where necessary, can always be put to use. But the most important thing here is that they do not leave the Maxwell–Lorentz equations invariant. As we have already noted, Newtonian mechanics has convincingly demonstrated that three-dimensional space is Euclidean and that time is absolute in all inertial frames of reference. The question presents itself: can investigation into electromagnetic phenomena allow us to make a further contribution to our knowledge of space and time? The answer is yes. How can we gain an insight into the structure of space and time? Above all, from examination of the propagation of the front of an electromagnetic wave or of the motion of a test particle with a velocity close to that of light.

We now write Maxwell's equations in free space

$$\operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \qquad \operatorname{div} \mathbf{H} = 0, \tag{1.2.2}$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div} \mathbf{E} = 0$$

and derive the equation for the front of an electromagnetic wave. Speaking about the front, we imagine that all the components of the field before it are zero, and behind some of them are nonzero. Therefore, if there is such thing as the front of a wave, then on it some of the field components are bound to have a discontinuity. If we take some surface that moves in space and define on it a field, then, by Maxwell's equations, we can find derivatives of the field both on the surface and on another surface that is at an infinitesimal distance from the first one. For a field to have discontinuities on a surface, it is necessary

that the surface obeyed some equation. Such surfaces are called characteristics. Accordingly, a discontinuity is only possible on characteristics. We will deduce, following Fock, the equation of a characteristic for Maxwell's equations. Suppose that on the surface

$$w(x, y, z, t) = ct - f(x, y, z) = 0 \tag{1.2.3}$$

we define an electromagnetic field. For example, the component E_x will be

$$E_x(x, y, z, t) = E_x\left(x, y, z, \frac{f}{c}\right) = E_x^0(x, y, z).$$

We now take a derivative of field components with respect to some coordinate on the surface (1.2.3)

$$\frac{\partial E_x}{\partial x} + \frac{1}{c}\frac{\partial E_x}{\partial t}\frac{\partial f}{\partial x} = \frac{\partial E_x^0}{\partial x}.$$

Similar relations can be obtained for other components of the electromagnetic field. We combine them to obtain

$$\operatorname{curl}\mathbf{H} + \frac{1}{c}\left[\operatorname{grad} f\frac{\partial \mathbf{H}}{\partial t}\right] = \operatorname{curl}\mathbf{H}^0, \tag{1.2.4}$$

$$\operatorname{div}\mathbf{H} + \frac{1}{c}\operatorname{grad} f\frac{\partial \mathbf{H}}{\partial t} = \operatorname{div}\mathbf{H}^0. \tag{1.2.5}$$

Likewise,

$$\operatorname{curl}\mathbf{E} + \frac{1}{c}\left[\operatorname{grad} f\frac{\partial \mathbf{E}}{\partial t}\right] = \operatorname{curl}\mathbf{E}^0, \tag{1.2.6}$$

$$\operatorname{div}\mathbf{E} + \frac{1}{c}\operatorname{grad} f\frac{\partial \mathbf{E}}{\partial t} = \operatorname{div}\mathbf{E}^0. \tag{1.2.7}$$

Using the field equations, we will find the scalar products of (1.2.4) and (1.2.6) by grad f

$$c(\operatorname{curl}\mathbf{H}^0\operatorname{grad} f) = \left(\operatorname{grad} f\frac{\partial \mathbf{E}}{\partial t}\right),$$

$$c(\operatorname{curl}\mathbf{E}^0\operatorname{grad} f) = -\left(\operatorname{grad} f\frac{\partial \mathbf{H}}{\partial t}\right).$$

Using again the field equations, we rewrite expressions (1.2.5) and (1.2.7)

$$c \operatorname{div} \mathbf{H}^0 = \operatorname{grad} f \frac{\partial \mathbf{H}}{\partial t},$$

$$c \operatorname{div} \mathbf{E}^0 = \operatorname{grad} f \frac{\partial \mathbf{E}}{\partial t}.$$

By comparing these expressions with the previous ones we obtain

$$\operatorname{div} \mathbf{E}^0 - (\operatorname{grad} f \operatorname{curl} \mathbf{H}^0) = 0,$$

$$\operatorname{div} \mathbf{H}^0 + (\operatorname{grad} f \operatorname{curl} \mathbf{E}^0) = 0.$$

We have thus found the conditions to be met by the given functions $\mathbf{E}^0$ and $\mathbf{H}^0$. Using Maxwell's equations, we will write expressions (1.2.4) and (1.2.6) as

$$\frac{\partial \mathbf{E}}{\partial t} + \left[\operatorname{grad} f \frac{\partial \mathbf{H}}{\partial t}\right] = c \operatorname{curl} \mathbf{H}^0,$$

$$-\frac{\partial \mathbf{H}}{\partial t} + \left[\operatorname{grad} f \frac{\partial \mathbf{E}}{\partial t}\right] = c \operatorname{curl} \mathbf{E}^0.$$

Multiplying these in a vector manner by grad f and using the earlier relations and the field equations, along with the well-known formula

$$[\mathbf{a}[\mathbf{bc}]] = \mathbf{b}(\mathbf{ac}) - \mathbf{c}(\mathbf{ab})$$

we will find the following equations

$$[1 - (\operatorname{grad} f)^2]\frac{\partial \mathbf{H}}{\partial t} =$$

$$= \frac{\partial \mathbf{H}^0}{\partial t} - \left(\operatorname{grad} f \frac{\partial \mathbf{H}^0}{\partial t}\right)\operatorname{grad} f + \left[\operatorname{grad} f \frac{\partial \mathbf{E}^0}{\partial t}\right],$$

$$[1 - (\operatorname{grad} f)^2]\frac{\partial \mathbf{E}}{\partial t} =$$

$$= \frac{\partial \mathbf{E}^0}{\partial t} - \left(\operatorname{grad} f \frac{\partial \mathbf{E}^0}{\partial t}\right)\operatorname{grad} f - \left[\operatorname{grad} f \frac{\partial \mathbf{H}^0}{\partial t}\right].$$

On the right we find known functions. If the factor $1 - (\operatorname{grad} f)^2$ is nonzero, these equations can be solved for the derivatives with respect to time of the field functions, and so, using the earlier formulas, we will obtain finite values for all the other first derivatives of the field

as well. The field on the surface (1.2.3) will thus be continuous. For the field to display a discontinuity on the surface it is required that the factor at the time derivatives be zero. This is provided by the condition

$$(\operatorname{grad} f)^2 = 1.$$

If we write the surface equation for the function $w(x, y, z, t)$, the previous equation for the front of an electromagnetic wave will become

$$\frac{1}{c^2}\left(\frac{\partial w}{\partial t}\right)^2 - \left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2 - \left(\frac{\partial w}{\partial z}\right)^2 = 0. \qquad (1.2.8)$$

If we change to arbitrary permissible coordinates, the equation will assume the form

$$g^{ik}\frac{\partial w}{\partial x^k}\frac{\partial w}{\partial x^i} = 0, \quad x^0 = ct, \quad i, k = 0, 1, 2, 3. \qquad (1.2.9)$$

The function $g_{ik}(x)$ in the front equation is a metric tensor and determines the structure of space-time. Since we have derived g_{ik} from (1.2.8) with the help of an arbitrary permissible transformation of the coordinates, this implies that there is always a Galilean system of coordinates, in which the metric tensor g_{ik} in entire space-time has the form

$$g_{00} = 1, \; g_{11} = -1, \; g_{22} = -1, \; g_{33} = -1, \qquad (1.2.10)$$
$$g_{ik} = 0, \text{ if } i \neq k.$$

In terms of the metric tensor, the squared distance between two close points in space-time can be written as follows

$$ds^2 = g_{ik}dx^i dx^k, \quad i, k = 0, 1, 2, 3. \qquad (1.2.11)$$

In the coordinate system where the tensor components are given by (1.2.10) we have

$$ds^2 = c^2 dT^2 - dX^2 - dY^2 - dZ^2.$$

Since the metric tensor assumes the values (1.2.10), the tensor of the curvature of this space-time R_{iklm} is zero (see definition (2.2.24)). Such a space-time is aid to be pseudo-Euclidean. We see thus that the Maxwell–Lorentz equations, which describe electromagnetic phenomena, revealed that space and time are one and that its geometry is pseudo-Euclidean. When electromagnetic phenomena were studied and even when the Maxwell–Lorentz equations were obtained it never occurred to anybody that they would cause an upheaval in our ideas of space and time. So the electromagnetic studies have led to a discovery of fundamental significance, the discovery that space and time are unity and that its geometry is pseudo-Euclidean. We owe this dis-

covery to Poincaré and Minkowski. Poincaré was the first to find that the quantity (later referred to as an interval)

$$c^2t^2 - x^2 - y^2 - z^2$$

is invariant with respect to the Lorentz group.

Minkowski [6] in 1908 presented his paper "Space and Time" to natural scientists and physicians, who at the time were also interested in those problems. He said: "Ladies and gentlemen! My views on space and time, which I now intend to set forth to you, have grown up from a physical experimental foundation. Their strength lies therein. Their tendency is radical. From now on space on its own and time on its own must become fiction, and only some kind of fusion of the two concepts should retain independence." He went on: "First I want to show how, proceeding from now-accepted mechanics, using purely mathematical arguments, one can arrive at new ideas of space and time". And further: "The equations of Newtonian mechanics demonstrate double invariance. They retain their form, first, when the spatial coordinate system on which they are based is subjected to any *change in position* (i.e., to a three-dimensional transformation or a rotation, or a translation – *A.L.*), and second, when the state of motion of the system undergoes a change, namely when this system is imparted some *uniform translational motion* (i.e., Galilean transformation – *A.L.*); the zero point of time also plays no role" (we have seen that Newton's equations are invariant under shifts in time – *A.L.*).

Still further: "Feeling prepared for a transition to the axioms of mechanics, we generally view the axioms of geometry as preestablished; therefore, these two invariances are perhaps rarely formulated together, so to speak, in one breath. Each of them constitutes a definite closed group of transformations of the differential equations of mechanics. The existence of the first group is regarded as the main feature of space. The second group is rather treated with contempt, so that then to ignore the fact that, physical phenomena never tell us whether or not the space that we deem to be stationary is in uniform translational motion. These two groups thus fare separately. Their disparate nature, it seems, prevented their marriage. But it is precisely the united complete group, as a whole, that provides food for the mind..."

"An attempt to step over the concept of space might in fact be regarded as an audacity of mathematical thinking. But after such a step, which is still unavoidable, for real understanding of group G_c (meaning a group in four-dimensional space already – *A.L.*) the term 'relativity postulate' for the invariance requirement with reference to group G_c seems to me to be all too feeble. Since the gist of the pos-

tulate is that phenomena only give to us a world of four dimensions in space and time but the projections of the world on space and time may be taken quite arbitrarily, I would rather refer to this statement as the 'postulate of the absolute world'."

Historically, although Lorentz's transformations had already been known since 1904, the year when he had published his work [1], neither Lorentz himself nor Einstein did admit the existence of a unified space-time, defined by one geometry. This was in essence done by Minkowski, who following Poincaré's reasoning, gained a complete understanding of this phenomenon. This crucial step to unite space and time into one whole and the introduction of appropriate geometry form is in essence, the content of the special theory of relativity. People, who generally have just a superficial knowledge of the theory, often think that this is its mathematical interpretation. No, this is precisely the content of the special theory of relativity.

Physical processes all occur in the four-dimensional world, i.e., in space and time, but the geometry of this world is pseudo-Euclidean. In the Euclidean world this is simply the Pythagoras theorem, which states that the distance between two points with the Cartesian coordinates (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) is given by

$$l_{12}^2 = (X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2.$$

In the four-dimensional world time and coordinates of any event (T, X, Y, Z) give us a world point. Therefore, in the four-dimensional world (space-time) we can introduce the notion of the distance between two world points, called the interval, but now in Cartesian coordinates it has a somewhat different form:

$$s_{12}^2 = c^2(T_1 - T_2)^2 - (X_1 - X_2)^2 - (Y_1 - Y_2)^2 - (Z_1 - Z_2)^2 = c^2(T_1 - T_2)^2 - l_{12}^2. \tag{1.2.12}$$

Since in (1.2.12) the spatial and temporal parts have different signs, Klein and Hilbert suggested to refer to space as pseudo-Euclidean.

Expression (1.2.12) for the interval does not follow from any more general principles. It is itself a fundamental principle of present-day physics, which states that space and time are a unity, and that its geometry is determined by the interval (1.2.12). An infinitesimal interval between two events

$$ds^2 = c^2dT^2 - dX^2 - dY^2 - dZ^2 \tag{1.2.13}$$

is an invariant in this four-dimensional world.

Minkowski understood that the essence of the theory of relativity, precisely of the special theory of relativity, is the fact that physical processes all occur in space-time whose geometry is pseudo-Eucli-

dean. At the time Einstein had no such an insight into the heart of the special theory of relativity. This is specifically attested by his opinion of the mathematical works on Relativity [7]: "Now that Relativity go assaulted by mathematicians I myself no longer understand it!" When the general theory of relativity came upon the scene, however, Einstein had already perceived the greatness of Minkowski's effort and praised his works [8]: "...without which the general theory of relativity... would perhaps remain in its infancy."

The term "theory of relativity" (or later "the special theory of relativity") is a misnomer. Historically, it emerged long ago, at the turn of the century, and so it is to stay, but its fundamental content should be clearly understood. The theory of relativity is not another foggy piece of philosophy on relativity. This is a theory of space-time. Investigations into various forms of matter, of its laws of motion, are at the same time investigations into space-time. Although the very structure of space-time was laid bare to us through our probings into matter. We sometimes speak about space-time as an arena upon which phenomena play out their parts. But we will not be in error if we remember that the arena does exist on its own, without matter.

It is sometimes said that space-time (Minkowski's world) has been given *a priori,* because its structure is unaffected by matter. Admittedly, the concepts of space-time that have emerged in natural sciences are a step in our knowledge of nature, but even then space-time is inseparable from matter and does not exist *a priori.*

We now return to the question of the relativity principle. Essentially it means: for all physical phenomena, and not only mechanical, that occur in some inertial frame of reference no physical experiments can determine whether the frame is at rest or moving with uniform velocity. This principle is a special manifestation of the fact that all physical events occur in space-time with the metric (1.2.12).

The constant c that enters into (1.2.12) is found experimentally (it is virtually the same as in Maxwell's equations). By and large, c is the limiting speed for the propagation of any interaction. We will be looking at this later in the book.

Since the geometry of space-time is determined by expression (1.2.13), and ds^2 is an invariant, whose magnitude is independent of the choice of the frame, we are led to conclude immediately that length is relative and time is relative.

Indeed, since the four-dimensional interval, involving the coordinates and time, is an invariant, the magnitude of a segment in a three-dimensional space will no longer be an invariant, and so in different frames of reference it will be different, with the result that the

notion of the length of a three-dimensional segment will not be absolute in nature. Hence the fundamental conclusion concerning the relativity of time lengths in various frames, due to Einstein [4].

It should be noted that not infrequently the concept of invariance is loosely used in contexts corresponding to the notions of covariance or form-invariance. We will, therefore, recall here the definition of these concepts and point out their differences.

An equation is said to be covariant under a coordinate transform, if its new unknown functions expressed in terms of new variables satisfy equations of the same form as the old functions in terms of old variables. The covariance requirement for equation is thus no reflection of any principle but rather a mathematical requirement.

Fock [9] found that for an equation to be covariant it was sufficient that under any permissible coordinate transformations it followed tensor laws. We will illustrate this by an example. The equations of relativistic mechanics

$$\frac{DU^i(x)}{Ds} = F^i(x) \tag{1.2.14}$$

are covariant, since in tensor algebra under an arbitrary permissible coordinate transformation

$$x'^i = x'^i(x^m) \tag{1.2.15}$$

the new functions in new variables $U'^i(x')$ will satisfy an equation of the same form as the original equation (1.2.14)

$$\frac{D'U'^i(x')}{D's} = F'^i(x'),$$

i.e., in transferring from coordinates x to coordinates x' the quantities in (1.2.14) are all replaced by the corresponding primed quantities.

It should be stressed that, generally speaking, the functional dependence of the metric tensor of space-time g'_{ni} on new coordinates under transformations (1.2.15) can change. It follows that if in the original frame of reference the metric tensor g_{ni} was one function of coordinates x, then in the primed form it may be a different function of coordinates x'. Covariant equations always include the metric tensor of space-time or its derivatives, therefore in the general case the functional dependence (functional form) of the covariant equations of new coordinates under transformations (1.2.15) will change.

To see this, we should take into account that under the coordinate transformations (1.2.15) the metric tensor of space-time is trans-

formed by

$$g'_{ni}(x') = \frac{\partial x^l}{\partial x'^n} \frac{\partial x^m}{\partial x'^i} g_{lm}(x(x')).$$

It is quite natural then that in new coordinates covariant equations do not retain their functional dependence, and so in different frames an event is described differently, i.e., in the general case events in different frames will occur differently.

Form-invariance for a metric under some coordinate transformations (i.e., that the functional dependence of the metric tensor under the transformation be unaltered) is a more stringent requirement than the covariance of equations. This requirement is a constraint on the class of frames of reference: they must be such that when transformed into one another the functional form of the metric tensor of space-time would remain unchanged.

More precisely, the functional dependence of g_{ni} on coordinates x in one frame of reference is the same as the dependence of g'_{ni} on coordinates x' in any other frame in the same class.

This requirement guarantees, however, that for the entire group of transformations that leave a metric form-invariant, the functional dependence of the field equations on new coordinates will remain unchanged. Therefore, in all frames such that any transformations between them leave the metric form-invariant all physical phenomena will proceed in the same manner, so that we will be in no position to establish in which of the frames we reside.

To sum up, covariance and form-invariance are different concepts. Transformations ensuring covariance of the field equations in the general case include transformations between various allowable frames of reference that are different in terms of physical phenomena description. In contrast, transformations that provide form-invariance of the metric tensor of space-time (and hence the form-invariance of covariant equations) include transformations only between physically equivalent frames of reference: in these frames all physical events occur similarly under appropriate initial and boundary conditions. Since the presence or absence of such transformations is wholly predetermined by the nature of the geometry of space-time, the special principle of relativity is essentially a trivial manifestation of the pseudo-Euclidean geometry of physical space-time.

Therefore any explanation or derivation of the principle from some special postulates does not reflect the essence of the problem. When formulating the special theory of relativity, such postulates, however, are viewed as the cornerstone of the theory. Specifically,

the postulate of the constancy of the speed of light is introduced on the grounds that light propagates in a vacuum with a constant velocity c, which is independent of the state of the motion of the body under consideration.

Further, using this statement as the basis we define the procedure of synchronizing clocks located at various points in space A and B, through an exchange of light signals. The procedure can be visualized as follows: from point A at some moment of time t_A by "A-clock" a light signal is sent to point B. It arrives at B at time t_B by "B-clock", is reflected there and returns to A at time t'_A by "A-clock". By definition, the clocks will be synchronized if

$$t_B - t_A = t'_A - t_B. \tag{1.2.16}$$

This relation is readily obtained using the postulate of the constancy of the speed of light. Indeed, if the distance between points A and B is denoted by R, then when the light signal arrives at point B the clock there synchronized with clock A will show

$$t_B = t_A + \frac{R}{c}.$$

Likewise, when the light signal returns to B the clock there must show

$$t'_A = t_B + \frac{R}{c}.$$

Subtracting the second equality from the first one, we will obtain the synchronization condition (1.2.16). From this condition

$$t_B = \frac{1}{2}(t_A + t'_A) = t_A + \epsilon(t'_A - t_A),$$

where $\epsilon = 1/2$.

Synchronization introduced in such a manner enables us to speak about "simultaneity" of events taking place at different points in space.

In our further arguments we will lean on the principle of relativity: it is required that the law of propagation of a spherical light wave be independent as to in which of the two coordinate systems that move relative to each other at a constant velocity the propagation process is described. Then, assuming that the spherical wave is emitted at $t = t' = 0$, when the origins of the frames happened to coincide, the velocity of light in one frame is

$$c^2 = \frac{x^2 + y^2 + z^2}{t^2}, \tag{1.2.17}$$

and in the other

$$c^2 = \frac{x'^2 + y'^2 + z'^2}{t'^2}, \tag{1.2.18}$$

where the coordinates (x, v, z, t) and (x', y', z', t') are related by the Lorentz transform. In the particular case of the propagation of light from the origin along the x-axis the formulas (1.2.17) and (1.2.18) yield

$$c = \frac{x}{t}, \quad c = \frac{x'}{t'} .$$

This construction of the theory of relativity is ambiguous and somewhat formal, since from the very beginning it does not provide for analysis of what the distance between two points in space is; what is more, the concept of time here is associated with some synchronization of clocks. The first to bring to light some of these questions was Reichenbach [10]. He put forward the following general condition for synchronization:

$$t_B = t_A + \epsilon(t'_A - t_A), \quad 0 < \epsilon < 1.$$

Einstein's synchronization condition holds at $\epsilon = 1/2$. Reichenbach's approach could now be presented as follows.

First, nothing provides for the velocity of light in one direction to be equal to that in the opposite direction. Suppose the velocity of light along the positive x-axis is c_1, in the opposite direction $-c_2$. Then at the moment the light signal arrives at B the time by the clock there must by

$$t_B = t_A + \frac{X_{AB}}{c_1}, \tag{1.2.19}$$

where X_{AB} is the distance between points A and B. Similarly, the signal must return to A by the "A-clock" at

$$t'_A = t_B - \frac{X_{AB}}{c_2}, \quad c_2 < 0. \tag{1.2.20}$$

Subtracting from (1.2.20) expression (1.2.19) gives

$$t'_A - t_B = t_B - t_A - X_{AB}\left(\frac{1}{c_1} + \frac{1}{c_2}\right).$$

Combining them gives

$$X_{AB} = (t'_A - t_A)\ \frac{c_1 c_2}{c_2 - c_1} .$$

Substituting this relation into the previous one, we arrive at

$$t_B = t_A + \epsilon(t'_A - t_A),$$

where $\epsilon = c_2/(c_2 - c_1)$. Since the velocities of light c_1 and c_2 may assume any constant values (and arbitrarily large at that), we obtain in the general case Reichenbach's synchronization

$$t_B = t_A + \epsilon(t'_A - t_A), \quad 0 < \epsilon < 1.$$

Consequently, arbitrariness in clock synchronization, first noted by Reichenbach, indicates that Einstein's approach to the construction of relativity theory is not the only one possible. This caused a confusion in the understanding of the essence of the theory of relativity, especially in sorting out the primary from the secondary. After Reichenbach, clock synchronization was discussed by Mandelshtam [11] and later by Tyapkin [12].

However, the opinion that the key point of the special theory of relativity is the concept of simultaneity is deeply erroneous. The concept of simultaneity is based on the clock synchronization procedure. But the synchronization, as it has been shown by Reichenbach, can be fairly arbitrary. It is a simple consequence of the choice of a frame or reference. Moreover, there exist frames (e.g., accelerated ones) in which clocks cannot be synchronized, although physical events in the frame can be described in terms of the special theory of relativity as well.

It follows from the foregoing that the postulate of the constancy of the speed of light (as defined by Einstein) does not hold good in the general case of arbitrary synchronization, thereby some questions arise.

How can physical time be determined when clocks are synchronized according to Reichenbach? How can the physical distance between two points in space be determined then? What is the limiting permissible velocity of signal propagation?

To answer these questions one should understand the essence of the theory of relativity. Otherwise it is impossible to give correct answers to these and other questions. It is precisely because of the lack of understanding of the principal element in the special theory of relativity that some books erroneously maintain that the special theory of relativity cannot be applied to describe events in noninertial frames of reference, that one should then rely on the general theory of relativity, and that some concepts in physics are conventions. It is to the clarification of these issues that this book is primarily devoted.

1.3. Lorentz Transformations

Newton's mechanics, as we now know, has estabished anew that the geometry of our three-dimensional world is Euclidean, and that time is absolute. Surprise came from electrodynamics: a serious analysis shows that space and time are married into one (four-dimensional) geometry and that this geometry is pseudo-Euclidean (with metric (1.2.10)). And given a geometry, we can in space introduce various frames of reference. We will think of the frame we have dealt with so far (ct and $\mathbf{r}$ – three Cartesian coordinates), which leads to metric (1.2.13), as a Galilean frame of reference. Generally speaking, however, physical events can be described in other frames as well, but still belonging to the class of the so-called permissible frames. A definition of permissible frames is given below.

We will now look at how the Lorentz coordinate transformations come about. Take the expression for the interval in a Galilean system of coordinates

$$ds^2 = c^2 dT^2 - dX^2 - dY^2 - dZ^2 \tag{1.3.1}$$

and subject it to Galilean transformations

$$x = X - vT, \quad t = T, \quad y = Y, \quad z = Z. \tag{1.3.2}$$

The inverse transformations are

$$X = x + vt, \quad T = t, \quad Y = y, \quad Z = z, \tag{1.3.3}$$

where X, Y, Z, T are Galilean coordinates.

We now take differential of both sides of (1.3.3) and substitute dT, dX, dY and dZ into expression (1.3.2) to obtain

$$ds^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) - 2v\, dx\, dt - dx^2 - dy^2 - dz^2. \tag{1.3.4}$$

Note that the right-hand side of (1.3.4) now contains the mixed term $dx\, dt$. We can get rid of it. To this end, we separate in (1.3.4) a complete square.

The interval (1.3.4) will take the form

$$ds^2 = c^2 \left[dt\sqrt{1 - \frac{v^2}{c^2}} - \frac{v}{c^2} \frac{dx}{\sqrt{1 - \frac{v^2}{c^2}}} \right]^2 - \frac{dx^2}{1 - \frac{v^2}{c^2}} - dy^2 - dz^2. \tag{1.3.5}$$

We thus see that expression (1.3.5) for the interval ds^2 consists of two parts, positive and negative. The positive part will have a time-like nature, and the negative one space-like.

We now introduce a new time

$$T' = t\sqrt{1 - \frac{v^2}{c^2}} - \frac{v}{c^2}\,\frac{x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.3.6}$$

and new coordinates

$$X' = \frac{x}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad Y' = y, \quad Z' = z. \tag{1.3.7}$$

In these variables expression (1.3.5) will then be exactly like in (1.2.13), only time and coordinate differentials will be primed:

$$ds^2 = c^2 dT'^2 - dX'^2 - dY'^2 - dZ'^2. \tag{1.3.8}$$

As a result, two successive transformations (1.3.2) and (1.3.6)–(1.3.7) will leave the metric form-invariant. Substituting expressions (1.3.2) into (1.3.6) and (1.3.7), we will obtain the well-known Lorentz transformations

$$T' = \frac{T - \frac{vX}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad X' = \frac{X - vT}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{1.3.9}$$
$$Y' = Y, \qquad Z' = Z.$$

The inverse transformations have the form

$$T = \frac{T' + \frac{vX'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad X = \frac{X' - vT'}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{1.3.10}$$
$$Y = Y', \qquad Z = Z'.$$

We have thus introduced a new concept of time (T') and a new coordinate (X'), so that the expression for the interval once again became

diagonal. It may well be asked why time and coordinates in expression (1.2.13) differ from those in (1.3.8), and what should be taken to be time and what a coordinate? We will look at this in more detail later in the book, and will only note now that all physical processes can be described in any permissible coordinates (t, x, y, z) and this description will be as full-fledged as, say, in the coordinates (T, X, Y, Z). But quantities (t, x, y, z) will then be coordinate quantities that are not directly related to physical quantities. We will illustrate this by a simple example. It is common knowledge that in three-dimensional space we can equally use the Cartesian coordinates (X, Y, Z), the cylindrical coordinates (r, φ, z), and the spherical coordinates (ρ, θ, φ).

However the squared differential of the distance between two close points will look differently in these coordinate systems:

$$d\mathbf{R}^2 = dX^2 + dY^2 + dZ^2,$$

$$d\mathbf{R}^2 = dr^2 + r^2 d\varphi^2 + dz^2,$$

$$d\mathbf{R}^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\varphi^2.$$

These include not only differentials of length, but, also, say, differentials of angles, and each time we have some function of appropriate coordinates, the so-called metric coefficients. And only the differentials with their metric coefficients can give us physical quantities. In the general case, the differentials of the four coordinates have themselves no physical meaning, i.e., they are directly connected neither with the distance between points in a three-dimensional space nor with the temporal course of processes.

We can now draw some conclusions.

Form-invariance. In the example above we took special care that the interval be form-invariant: we first substituted into ds^2 the transformations (1.3.3) of old coordinates to new ones, then we chose in the moving frame of reference a new set of variables (1.3.6)–(1.3.7) with the result that the interval in the primed frame had perfectly the same form as the one in the nonprimed one. In a pseudo-Euclidean geometry, an interval is always form-invariant, but in more complicated geometries this is not always the case. Since under transformations that leave a metric form-invariant all the equations of physics (mechanics, electrodynamics, and so on) remain form-invariant, the functional dependence of the field equations on new coordinates will remain unchanged. Therefore, in all frames in which any transformations leave a metric form-invariant all physical phenomena described by these equations will occur in the same manner, and so no experiment will tell us in which of the frames we reside. Accordingly, the form-invariance of the interval (1.3.1) when the transformations (1.3.3), (1.3.6)

and (1.3.7) are applied in succession guarantee that all the frames of reference connected by the resultant transformation (1.3.9) are physically identical; that is, no physical experiment will distinguish one frame from another.

Galileo's principle of relativity is thus a special consequence of the fact that the geometry of space-time is pseudo-Euclidean. As we have seen above the Maxwell–Lorentz equations definitely suggest that space-time is one and its geometry is pseudo-Euclidean. This statement is a strict mathematical proof. Electromagnetic studies have thus lead to a revelation of the structure of space-time. This, in turn, suggested the hypothesis that for other physical processes as well space-time is pseudo-Euclidean.

Therefore, at the heart of Relativity (the special theory of relativity) lies the following (this is a postulate): physical processes occur in a four-dimensional space (ct and spatial coordinates) whose geometry is pseudo-Euclidean. The principle of relativity is merely a special manifestation of this fundamental postulate.

Absolute value of interval. It should be stressed once more that if we take the interval (1.3.1), which is an invariant, then the distance between two points in a three-dimensional space and the time between two events are no absolute concepts *per se* (as was the case in Newtonian mechanics). The special theory of relativity has divested them of their absolutism, and made them relative concepts. It is the interval that is absolute. It can be positively or negatively defined, and for a light beam it is zero. If we term $ds^2 > 0$ the time-like interval, $ds^2 < 0$ space-like interval, $ds^2 = 0$ light (isotropic) interval. then these concepts are absolute too. And so all sorts of coordinate transformations cannot violate this absolutism.

We have earlier found formulas of the Lorentz transformations for the case where one frame of reference move relative to another with a constant velocity along the X-axis. Before we leave this section we will find transformation formulas for arbitrary constant velocity $\mathbf{v}$.

We denote by X, Y, Z the components of $\mathbf{R}$, and carry out the Galilean transformation

$$\mathbf{r} = \mathbf{R} - \mathbf{v}T, \quad t = T.$$

In terms of these variables the expression for (1.2.11) becomes

$$ds^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) - 2\mathbf{v}\, d\mathbf{r}\; dt - d\mathbf{r}^2.$$

We introduce the notation

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

We can then write the expression for the interval as

$$ds^2 = c^2 \left[\frac{1}{\gamma} dt - \frac{\gamma}{c^2} \mathbf{v} d\mathbf{r} \right]^2 - d\mathbf{r}^2 - \frac{\gamma^2}{c^2} (\mathbf{v} d\mathbf{r})^2 .$$

Our principal aim is to find the new variables T' and $\mathbf{R}'$ such that they would enable us to write the above expression in diagonal form

$$ds^2 = c^2 (dT')^2 - (d\mathbf{R}')^2 .$$

We introduce a new time

$$T' = t \frac{1}{\gamma} - \frac{\gamma}{c^2} \mathbf{v}\mathbf{r}.$$

Expressing the right-hand side using Galilean transformations through T and $\mathbf{R}$, we will obtain

$$T' = \gamma \left(T - \frac{\mathbf{v}\mathbf{R}}{c^2} \right).$$

We will also express in terms of T and $\mathbf{R}$ the negative part of the interval

$$(d\mathbf{r})^2 + \frac{\gamma^2}{c^2} (\mathbf{v} d\mathbf{r})^2 = (d\mathbf{R})^2 + \frac{\gamma^2}{c^2} (\mathbf{v} d\mathbf{R})^2$$

$$- 2\gamma^2 \mathbf{v} d\mathbf{R} dT + \gamma^2 v^2 (dT)^2 .$$

It is easily seen that the first two terms can be written as the square of a vector:

$$(d\mathbf{R})^2 + \frac{\gamma^2}{c^2} (\mathbf{v} d\mathbf{R})^2 = \left[d\mathbf{R} + (\gamma - 1) \frac{\mathbf{v}(\mathbf{v} d\mathbf{R})}{v^2} \right]^2 .$$

Hence

$$(d\mathbf{r})^2 + \frac{\gamma^2}{c^2} (\mathbf{v} d\mathbf{r})^2 =$$

$$= \left[d\mathbf{R} + (\gamma - 1) \frac{\mathbf{v}(\mathbf{v} d\mathbf{R})}{v^2} \right]^2 - 2\gamma^2 \mathbf{v} d\mathbf{R} dT + \gamma^2 v^2 (dT)^2 ,$$

but the right-hand side is the square of the vector

$$(d\mathbf{r})^2 + \frac{\gamma^2}{c^2} (\mathbf{v} d\mathbf{r})^2 = \left[d\mathbf{R} + (\gamma - 1) \frac{\mathbf{v}(\mathbf{v} d\mathbf{R})}{v^2} - \gamma \mathbf{v}\, dT \right]^2 .$$

The space-like part of the interval thus assumes the diagonal form

$$(d\mathbf{r})^2 + \frac{\gamma^2}{c^2}(\mathbf{v}d\mathbf{r})^2 = (d\mathbf{R}')^2,$$

where

$$\mathbf{R}' = \mathbf{R} + (\gamma - 1)\frac{\mathbf{v}(\mathbf{vR})}{v^2} - \gamma\mathbf{v}T.$$

We have thus derived transformation formulas for coordinates and time in the general case of motion with a uniform speed $\mathbf{v}$

$$T' = \gamma\left(T - \frac{(\mathbf{vR})}{c^2}\right), \quad \mathbf{R}' = \mathbf{R} + (\gamma - 1)\frac{\mathbf{v}(\mathbf{vR})}{v^2} - \gamma\mathbf{v}T. \qquad (1.3.11)$$

Using the Lorentz transformations (1.3.9) we can readily deduce formulas for velocity composition. To this end, we differentiate (1.3.9) with respect to T

$$\frac{dT'}{dT} = \frac{1 - \frac{vV_x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad V'_x\frac{dT'}{dT} = \frac{V_x - v}{\sqrt{1 - \frac{v^2}{c^2}}},$$

$$V'_y\frac{dT'}{dT} = V_y, \quad V'_z\frac{dT'}{dT} = V_z.$$

Substituting the first of these into the subsequent ones gives

$$V'_x = \frac{V_x - v}{1 - \frac{vV_x}{c^2}}, \quad V'_y = \frac{V_y\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vV_x}{c^2}},$$

$$V'_z = \frac{V_z\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vV_x}{c^2}}. \qquad (1.3.12)$$

These relations were first derived in Poincaré [3].

In a similar manner we can derive velocity composition formulas in the general case. For this purpose, we differentiate formulas

(1.3.11) with respect to T

$$\frac{dT'}{dT} = \gamma\left(1 - \frac{\mathbf{vV}}{c^2}\right),$$

$$\mathbf{V}'\frac{dT'}{dT} = \mathbf{V} + (\gamma - 1)\,\frac{\mathbf{v}(\mathbf{vV})}{v^2} - \gamma\mathbf{v}.$$

Substituting the first of these into the second one, we will find

$$\mathbf{V}' = \frac{\mathbf{V} + (\gamma - 1)\,\dfrac{\mathbf{v}(\mathbf{vV})}{v^2} - \gamma\mathbf{v}}{\gamma\left[1 - \dfrac{\mathbf{vV}}{c^2}\right]}\,. \tag{1.3.13}$$

We will now show that the velocity of one body relative to another one is always less than the speed of light. Suppose that in an inertial frame of reference one body moves at $\mathbf{v}$ and another at $\mathbf{V}$. Take another frame of reference such that the first body's velocity in it will be zero, then in this frame the velocity of the second body will be $\mathbf{V}'$. Velocity $\mathbf{V}'$ will be the relative velocity of the second body in relation to the first one. We can see this readily by using the formula

$$V'^2 = \frac{(\mathbf{v} - \mathbf{V})^2 - \dfrac{1}{c^2}\,[\mathbf{vV}]^2}{\left(1 - \dfrac{\mathbf{vV}}{c^2}\right)^2}\,. \tag{1.3.14}$$

From this we obtain the relation

$$1 - \frac{V'^2}{c^2} = \frac{\left(1 - \dfrac{v^2}{c^2}\right)\left(1 - \dfrac{V^2}{c^2}\right)}{\left(1 - \dfrac{\mathbf{vV}}{c^2}\right)^2}\,. \tag{1.3.15}$$

It follows, in particular, that if

$$v^2 < c^2 \quad \text{and} \quad V^2 < c^2,$$

then the relative velocity V' will always be less than the speed of light. If the relative velocity of two bodies is infinitesimal, i.e.,

$$\mathbf{v} = \mathbf{V} + d\mathbf{V},$$

then we can obtain from (1.3.14) the square of the distance between

two close points in the space of velocities

$$ds^2 = \frac{(c^2 - v^2)(d\mathbf{V})^2 + (\mathbf{V}\, d\mathbf{V})^2}{(c^2 - v^2)^2} c^2. \tag{1.3.16}$$

The resultant space of velocities is the Lobachevsky space. All the properties of this space are defined by (1.3.16).

1.4. Relativity of Time and Length Contraction

Consider the time-like interval

$$ds^2 = c^2 dT^2 - dX^2 - dY^2 - dZ^2 = c^2 dT'^2 - dX'^2 - dY'^2 - dZ'^2 > 0.$$

We can write this in the form

$$c^2 dT^2 - d\mathbf{R}^2 = c^2 dT'^2 - d\mathbf{R}'^2 > 0.$$

Since this interval is more than zero, there exists a frame of reference (say, the primed one) in which two close events occur at one point in space ($d\mathbf{R}' = 0$). The space-time interval is then reduced to the difference of times alone in the primed system:

$$c^2 dT'^2 = c^2 dT^2 \left[1 - \frac{1}{c^2} \left(\frac{d\mathbf{R}}{dT} \right)^2 \right] = c^2 dT^2 \left[1 - \frac{\mathbf{v}^2(T)}{c^2} \right],$$

where we have introduced the velocity $\mathbf{v}(T) = d\mathbf{R}/dT$. We can now determine how a change in time in the primed frame is related to the corresponding change in time in the unprimed frame for a process localized in the primed frame:

$$dT' = dT \sqrt{1 - \frac{\mathbf{v}^2(T)}{c^2}}, \qquad T_2' - T_1' = \int_{T_1}^{T_2} \sqrt{1 - \frac{\mathbf{v}^2(T)}{c^2}}\, dT. \tag{1.4.1}$$

This expression is due to Einstein, it is the manifestation of the relativity of time.

We should admit that Lorentz, having discovered his celebrated transformations, failed to perceive their significance, and so the next decisive step to the special theory of relativity was made independently by Poincaré and Einstein, although they went different paths. The former subjected to mathematical analysis the group properties of the four-dimensional space, the latter subjected to operation analysis the

relativity of time. On the subject Pauli noted at the conference devoted to the 50th anniversary of Relativity [13]: "Both Einstein and Poincaré have drawn on the preparatory work of H.A. Lorentz, who came quite close to the final result, but could not make the last decisive step. I see in the coincidence of the results arrived at independently by Einstein and Poincaré a deep sense of the harmony of the mathematical method and analysis performed using thought experiments resting on the entire body of physical experimental evidence."

But the final formulation and the most profound understanding of the fact that we here deal with a unified geometry of space-time in terms of Poincaré is due to Minkowski. Poincaré and Minkowski discovered the geometry of space-time, known as pseudo-Euclidean geometry.

Let us take another example. Suppose that the interval between two events is space-like, i.e., that $ds^2 < 0$. Then, there exists a frame of reference in which the two events are simultaneous, i.e., $dT' = 0$. If the events take place at point lying on the X-axis, the space-time interval will be

$$ds^2 = -dX'^2, \tag{1.4.2}$$

i.e., the picture reduces to purely spatial distance. In any other frame of reference we have

$$ds^2 = c^2 dT^2 - dX^2. \tag{1.4.3}$$

Introducing the notation for the lengths of segments connecting the points where the events are taking place

$$dl_0^2 = dX^2, \quad dl^2 = dX'^2,$$

and equating the expressions (1.4.2) and (1.4.3), we will get

$$c^2 dT^2 + dl^2 = dl_0^2. \tag{1.4.4}$$

It follows immediately from this that the length dl in the primed frame of reference is smaller than dl_0 in the unprimed frame, i.e., $dl < dl_0$. Using the inverse Lorentz transformations (1.3.10) we will find that

$$dT = \frac{dT' + \frac{v}{c^2} dX'}{\sqrt{1 - v^2/c^2}}. \tag{1.4.5}$$

Since $dT' = 0$, substitution of (1.4.5) into the relation (1.4.4) gives

$$dl = dl_0\sqrt{1 - v^2/c^2}. \tag{1.4.6}$$

We see that this length contraction is nothing else but a consequence of (a) the structure of the geometry of four-dimensional time-space and (b) the method of measurement of the length of a moving segment.

It is important to note that this length contraction, unlike the generally held viewpoint, is no Lorentz–Fitzgerald contraction.

The examples just discussed indicate that the interval in the four-dimensional world is not something abstract, that it is quite measurable. If an interval is time-like, we can always chose a suitable frame of reference and measure it using a clock alone. If an interval is space-like, we can, having chosen an appropriate frame of reference, measure it with the help of a ruler.

Let us now take a closer look at the measurement of lengths and times in inertial frames of reference. To begin with, we notice that in each frame there is its own concept of length, which by no means supposes that the coordinates of the beginning and end of the segment are measured at the same time. This is manifested in the most graphic manner when an observer with a ruler is at rest relative to the segment to be measured. He then may at a moment of time T_1 look at the beginning of the segment and establish that it corresponds to the mark X_1 on the ruler, and at time $T_2 > T_1$ he can register that the end of the segment coincides with the mark $X_2 > X_1$ on the ruler. And although the indications X_1 and X_2 were taken not at the same time ($T_2 - T_1 > 0$) the length of the segment will nevertheless be

$$l = X_2 - X_1.$$

Suppose now that we have two inertial frames of reference moving relative to one another with a velocity V, and we will have the convention that one of them is called primed, the other unprimed. Further, we select in these frames the coordinates (X, Y, Z, T) and (X', Y', Z', T') such that, first, a metric in them would be (1.3.1) and (1.3.8) respectively; second, the coordinate axes in the frames would be pointing in the same direction. Without loss of generality, we will assume also that the relative velocity V is directed along the X-axis. The coordinates and time in these frames will then be related by the Lorentz transformations (1.3.9) and (1.3.10).

We now consider some two events with coordinates (X_1, Y_1, Z_1, T_1) and (X_2, Y_1, Z_1, T_2). To be more specific, we will suppose that $X_2 > X_1$, $T_2 > T_1$. Using the Lorentz transformations (1.3.9),

we can easily find that in the primed frame of reference

$$X'_2 - X'_1 = \frac{X_2 - X_1 - V(T_2 - T_1)}{\sqrt{1 - V^2/c^2}} ,$$

$$T'_2 - T'_1 = \frac{T_2 - T_1 - \dfrac{V}{c^2}(X_2 - X_1)}{\sqrt{1 - V^2/c}} .$$

Introducing the notation

$$w = \frac{X_2 - X_1}{T_2 - T_1} \geqslant 0, \tag{1.4.7}$$

we can then rewrite these expressions in the form

$$X'_2 - X'_1 = (X_2 - X_1)\frac{1 - V/w}{\sqrt{1 - V^2/c^2}} , \tag{1.4.8}$$

$$T'_2 - T'_1 = (T_2 - T_1)\frac{1 - Vw/c^2}{\sqrt{1 - V^2/c^2}} .$$

The interval between the events will be

$$s_{12}^2 = c^2(T_2 - T_1)^2 - (X_2 - X_1)^2. \tag{1.4.9}$$

Consider two cases:

(a) Let $s_{12}^2 < 0$. The point (X_2, Y_1, Z_1, T_2) then lies outside the light cone and $w > c$ from relations (1.4.7) and (1.4.9). Then $1 - V/w > 0$, and if $X_2 - X_1 > 0$, it follows from the first of (1.4.8) that $X'_2 - X'_1 > 0$. But although the quantity $X'_2 - X'_1$ has a fixed sign, the length of the segment $l' = X'_2 - X'_1$ in the primed frame of reference is not bound to contract, it may even be increased as compared with the length of the segment in the unprimed frame of reference.

In particular, it follows from formulas (1.4.8) that for two events for which in the unprimed frame we have

$$w > \frac{c^2}{V}\left(1 + \sqrt{1 - V^2/c^2}\right),$$

the length of a segment of the primed frame will always be larger than in the unprimed frame. For intervals for which

$$\frac{w}{c} \gg 1,$$

we have

$$(X_2' - X_1') = \frac{(X_2 - X_1)}{\sqrt{1 - V^2/c^2}} .$$

For two other events such that

$$c < w < \frac{c^2}{V} (1 + \sqrt{1 - V^2/c^2}),$$

the length of the segment in the primed frame will always be smaller than in the unprimed frame.

The lengths of segments of different space-like intervals lying along the X-axis in the frame K behave differently from the viewpoint of the system K: some become smaller, others larger. Statements that the length of a moving rod is contracted are erroneous.

Notice that for two events for which

$$w = \frac{1 + \sqrt{1 - V^2/c^2}}{V} c^2,$$

the length of the segment will not change, and the time interval will change its sign. When we speak about length contraction, we should always have in mind that we compare the lengths of a rod in some frame of reference with the length measured in another frame at the same time. It is for such a measurement that we can speak about length contraction. Such a way of measuring means specifying the interval

$$s_{12}^2 = -l^2 .$$

Length contraction is thus determined not only by the properties of space-time but also our measurement technique; therefore, unlike the slowing of the time of a localized process, it has no such physical significance.

The quantity $T_2' - T_1'$ in this case will have a fixed sign, even if $T_2 - T_1 > 0$:

$$T_2' - T_1' \begin{cases} > 0 \text{ for } V < \dfrac{c^2}{w}, \\ < 0 \text{ for } V > \dfrac{c^2}{w}, \\ = 0 \text{ for } V = \dfrac{c^2}{w}. \end{cases}$$

Accordingly, among the many frames of reference there exists one frame ($V = c^2/w$) in which events occur simultaneously, and the interval will have the form

$$s_{12}^2 = -l'^2 .$$

In this case, from expressions (1.4.8) we obtain the expression we have already obtained above:

$$X_2'^2 - X_1' = (X_2 - X_1)\sqrt{1 - V^2/c^2} .$$

(b) Let $s_{12}^2 > 0$. Point (X_2, Y_1, Z_1, T_2) will then lie within the light cone and $w < c$, therefore if $T_2 - T_1 > 0$, then in all primed frames of reference the time interval $T_2' - T_1'$ by virtue of the second of (1.4.8) appears to be strictly more than zero. The time interval $T_2' - T_1'$ in this case may be both shorter or longer than that in an unprimed frame of reference. Indeed, if we have

$$\frac{c^2}{V}\left(1 - \sqrt{1 - \frac{V^2}{c^2}}\right) < w < c,$$

then the time interval between events in the primed frame of reference will be shorter than that in the unprimed. But when

$$w < \frac{c^2}{V}\left(1 - \sqrt{1 - \frac{V^2}{c^2}}\right),$$

conversely, the time interval in the primed frame will be longer than in the unprimed one. For example, if $w \ll c$, then

$$T_2' - T_1' = \frac{T_2 - T_1}{\sqrt{1 - V^2/c^2}} .$$

The sign of the quantity $X_2' - X_1'$ will depend on the sign of $1 - V/w$

$$X_2' - X_1' \begin{cases} > 0 \text{ if } V < w, \\ < 0 \text{ if } V > w, \\ = 0 \text{ if } V = w. \end{cases}$$

Therefore, among the set of primed frames of reference there exists a frame ($V = w$) in which two events occur at the same point, i.e., the process is localized in this system. The interval in the primed system will then have the form

$$s_{12}^2 = c^2(T_2' - T_1')$$

and the time intervals between the events in the primed and unprimed

frames of reference will be related by

$$T_2' - T_1' = (T_2 - T_1)\sqrt{1 - V^2/c^2}.$$

This slowing of clocks is important in nature. It is because of this that we can create (and do create) high-energy particle beams with a lifetime of 10^{-8} s and transport them to experimental installations over hundreds of metres, although according to conventional picture, even if they moved at near-light velocities, they should flight through no more than 4 m.

So, if a particle at rest has a lifetime τ_0, then when it travels at a velocity V, it covers in a laboratory frame of reference the distance

$$L = \frac{V\tau_0}{\sqrt{1 - V^2/c^2}}.$$

It is often said that time flows at a slower rate in a moving inertial system than in a stationary one. This statement is not true, because from the viewpoint of the principle of relativity any inertial frame can be treated as stationary or moving. In this case, we mean not the flow of time in different frames but description in different inertial systems of a process localized in the primed frame of reference ($X_1' = X_2'$). It was in a system where the particle was at rest that spatial localization took place. In any other inertial frame of reference the acts of production and decay of a particle occur at different points in space. On the other hand, the presence of the spatial part of the interval leads (due to its invariance) to longer lifetimes of the particle in the laboratory reference frame.

We have seen that if in an inertial frame of reference two events took place at some point in space at different moments of time, then in another inertial frame these events occur at different points of space. In a similar manner, if in an inertial frame two events occurred simultaneously but at different points of space, then in another inertial frame these events will no longer be simultaneous.

An event at some point of space and at some moment of time is absolute. For example, if two particles collided at some instant, this event in any inertial frame of reference would be at one place and at one time.

We note that in classical mechanics any two events that occur at two different points and at different times in one inertial reference frame will occur at one place in another frame, if the latter, according to Galilean transformations, moves at a velocity

$$V = \frac{X_2 - X_1}{t_2 - t_1}.$$

In classical mechanics this is always possible, since there are no constraints on the magnitude of velocity. In relativistic theory this is only possible if the events are connected by the time-like interval

$$(X_2 - X_1)^2 < c^2(t_2 - t_1)^2,$$

i.e., when

$$\frac{V^2}{c^2} < 1.$$

In relativistic theory intervals between events are broadly classed as isotropic, space-like, and time-like. In classical physics there is no such division. It only has one class of events.

So far we have dealt with diagonal, i.e., Galilean, systems, where the metric has the form (1.2.13). We now make next step in the direction of generalization and consider an arbitrary system of coordinates in a pseudo-Euclidean space-time

$$ds^2 = g_{ik}\, dx^i dx^k, \quad i, k = 0, 1, 2, 3. \tag{1.4.10}$$

Here $x^0 = ct$, and x^1, x^2, x^3 are spatial coordinates, and g_{ik} is the metric tensor. It follows from the form of (1.4.10) that it is symmetric, i.e.,

$$g_{ik} = g_{ki}.$$

Coordinates x^i in (4.10) are now devoid of any physical meaning, and so, generally speaking, physical quantities have to be constructed out of them and the metric tensor of space-time.

We will now write out the expressions (1.4.10) and single out the zero indices

$$ds^2 = c^2 g_{00} dt^2 + 2c g_{0\alpha}\, dx^\alpha dt + g_{\alpha\beta}\, dx^\alpha dx^\beta,$$

$$\alpha, \beta = 1, 2, 3.$$

We will now do what we have made earlier when discussing Galilean transformations: we will construct a positive quantity on the basis of the first two terms. To do so, we will add and subtract the square of the quantity

$$\frac{g_{0\alpha}\, dx^\alpha}{\sqrt{g_{00}}}.$$

Then

$$ds^2 = c^2 \left[\sqrt{g_{00}}\, dt + \frac{g_{0\alpha}\, dx^\alpha}{c\sqrt{g_{00}}}\right]^2 - \left[-g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}}\right] dx^\alpha dx^\beta. \tag{1.4.11}$$

In arbitrary frames of reference we will thus obtain a counterpart of time

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0\alpha}\, dx^\alpha}{c\sqrt{g_{00}}} . \tag{1.4.12}$$

And this "time" will be physical.

The second term in (1.4.11) is nothing else but the squared distance between two points in conventional three-dimensional space

$$dl^2 = \kappa_{\alpha\beta}\, dx^\alpha dx^\beta, \tag{1.4.13}$$

where we have introduced a new tensor

$$\kappa_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}\, g_{0\beta}}{g_{00}} . \tag{1.4.14}$$

We have carried out this exercise for the pseudo-Euclidean metric; there was no transition to another geometry, just a change of coordinates. Changing a frame of reference does not change the geometry. Anticipating events I will say that the geometry is given by the Riemann tensor of tank four, known as the curvature tensor. For the pseudo-Euclidean geometry the Riemann tensor is zero in any permissible reference frame. It is to be noted that in the general case expression (1.4.12) is not a total differential, since it contains components of a metric tensor, which is generally some function of coordinates and time and not always meets the conditions

$$\frac{\partial}{\partial x^\alpha}\sqrt{g_{00}} = \frac{\partial}{\partial t}\, \frac{g_{0\alpha}}{c\sqrt{g_{00}}} ,$$

$$\frac{\partial}{\partial x^\beta}\, \frac{g_{0\alpha}}{\sqrt{g_{00}}} = \frac{\partial}{\partial x^\alpha}\, \frac{g_{0\beta}}{\sqrt{g_{00}}} , \tag{1.4.15}$$

which are necessary and sufficient for the right-hand side of (1.4.12) to be a total differential.

Therefore, although the interval of the type (1.4.10) in an arbitrary permissible frame can be represented in the form

$$ds^2 = (df^0)^2 - (df^1)^2 - (df^2)^2 - (df^3)^2 ,$$

the quantities $df^i(x^0, x^1, x^2, x^3)$ in the general case will not be total differentials. When the quantities $df^i(x^0, x^1, x^2, x^3)$ are total differentials

$$df^i = \frac{\partial f^i}{\partial x^k}\, dx^k ,$$

space-time has a pseudo-Euclidean geometry. Metric coefficients will then be

$$g_{ik} = \frac{\partial f^0}{\partial x^i}\frac{\partial f^0}{\partial x^k} - \frac{\partial f^1}{\partial x^i}\frac{\partial f^1}{\partial x^k} - \frac{\partial f^2}{\partial x^i}\frac{\partial f^2}{\partial x^k} - \frac{\partial f^3}{\partial x^i}\frac{\partial f^3}{\partial x^k}.$$

They are expressed in terms of arbitrary functions f^0, f^1, f^2, f^3. Using expressions for g_{ik}, we obtain

$$ds^2 = g_{ik}\, dx^i dx^k$$
$$= \left(\frac{\partial f^0}{\partial x^i}\, dx^i\right)^2 - \left(\frac{\partial f^1}{\partial x^i}\, dx^i\right)^2 - \left(\frac{\partial f^2}{\partial x^i} dx^i\right)^2 - \left(\frac{\partial f^3}{\partial x^i} dx^i\right)^2.$$

To find the general form of metric coefficients in an inertial reference frame we will chose in a three-dimensional space some arbitrary coordinates using the transformations

$$x'^{\alpha} = f^{\alpha}(x^1, x^2, x^3) \tag{1.4.16}$$

and introduce a new time

$$x'^{0} = f^0(x^0, x^1, x^2, x^3). \tag{1.4.17}$$

Transformations (1.4.16) and (1.4.17) do not bring us beyond the original inertial frame, and we will refer to them as permissible transformations in an inertial frame of reference. The metric coefficients in an inertial frame will then assume the most general form

$$g_{00} = \left(\frac{\partial f^0}{\partial x^0}\right)^2, \quad g_{0\alpha} = \frac{\partial f^0}{\partial x^0}\frac{\partial f^0}{\partial x^\alpha},$$
$$g_{\alpha\beta} = \frac{\partial f^0}{\partial x^\alpha}\frac{\partial f^0}{\partial x^\beta} - \sum_{\nu=1} \frac{\partial f^\nu}{\partial x^\alpha}\frac{\partial f^\nu}{\partial x^\beta}.$$

Therefore, in inertial frames the metric tensor $\kappa_{\alpha\beta}$ will have the following structure

$$\kappa_{\alpha\beta} = \sum_{\nu=1}^{3} \frac{\partial f^\nu}{\partial x^\alpha}\frac{\partial f^\nu}{\partial x^\beta}.$$

It follows that the components of the physical velocity in inertial frames have the form

$$V^\alpha = \frac{\partial f^\alpha}{\partial x^\beta}\frac{dx^\beta}{d\tau}.$$

The quantity $d\tau$ is a total differential. But in any noninertial frame of reference the quantity $d\tau$ will no longer be a total differential.

Using the previous expression, we will have

$$V = \frac{dl}{d\tau}, \quad dl^2 = \kappa_{\alpha\beta}\, dx^\alpha dx^\beta. \tag{1.4.18}$$

In inertial frames and in the absence of forces the motion of a material point is rectilinear and uniform

$$x'^\alpha = x_0'^\alpha + V^\alpha(\tau - \tau_0).$$

The quantities $d\tau$ and dl^2 given by formulas (1.4.12) and (1.4.13) were termed physical, because they do not depend on the choice of a coordinate system in a given inertial reference frame, since they are invariant under the transformations (1.4.16) and (1.4.17).

1.5. Invariance of Maxwell–Lorentz Equations and Law of Transformation of Electromagnetic Field

We have established above that electrodynamics has uncovered the unity of space and time and suggested that the geometry of space-time is pseudo-Euclidean, and its metric tensor in Galilean coordinates is

$$g_{00} = 1, \quad g_{11} = -1, \quad g_{22} = -1; \quad g_{33} = -1, \; g_{ik} = 0, \; \text{if } i \neq k. \tag{1.5.1}$$

Coordinate transformations that leave metric coefficients unchanged are called the motion of metric. Coordinate transformations that leave the metric form-invariant will be Lorentz transformations:

$$x'^i = a^i_k x^k, \quad i, k = 0, 1, 2, 3.$$

We take the differential of the left- and right-hand sides

$$dx'^i = d^i_k\, dx^k. \tag{1.5.2}$$

Throughout the book the repeated indices mean summation. The system of functions that can be transformed as a differential is known as the contravariant four-dimensional vector (four-vector)

$$A'^i = \frac{\partial x'^i}{\partial x^k} A^k. \tag{1.5.3}$$

We introduce the concept of the covariant vector. Recall that the simplest object is the field of a scalar that in any frame of reference is defined by one function and transformed as follows:

$$\Psi'(x') = \Psi(x(x')).$$

The gradient of a scalar function is transformed following the law

$$\frac{\partial\Psi'(x')}{\partial x'^{i}} = \frac{\partial x^{k}}{\partial x'^{i}} \frac{\partial\Psi(x)}{\partial x^{k}} .$$

A system of functions that is transformed as the gradient of a scalar function is said to be the covariant vector

$$A_i' = \frac{\partial x^k}{\partial x'^{i}} A_k. \tag{1.5.4}$$

The product of a covariant vector by a covariant vector is an invariant under the coordinate transformations

$$A_i' B'^{i} = \frac{\partial x^k}{\partial x'^{i}} \frac{\partial x'^{i}}{\partial x^l} A_k B^l = A_k B^k.$$

A covariant and a contravariant vectors are connected by

$$A_i = g_{ik} A^k, \quad A^i = g^{ik} A_k. \tag{1.5.5}$$

Hence

$$g_{ik}\, g^{kl} = \delta^l_k, \quad \delta^l_k = 1 \text{ at } l = k, \qquad \delta^l_k = 0 \text{ for } l \neq k. \tag{1.5.6}$$

We now introduce the concept of the tensor of rank two as a simple generalization of the vector. The contravariant tensor of the second rank is

$$A'^{ik} = \frac{\partial x'^{i}}{\partial x^l} \frac{\partial x'^{k}}{\partial x^m} A^{lm}. \tag{1.5.7}$$

The covariant tensor of the second rank is

$$A'_{ik} = \frac{\partial x^l}{\partial x'^{i}} \frac{\partial x^m}{\partial x'^{k}} A_{lm}. \tag{1.5.8}$$

The mixed tensor of the second rank is

$$A'^{i}_{k} = \frac{\partial x'^{i}}{\partial x^l} \frac{\partial x^m}{\partial x'^{k}} A^l_m. \tag{1.5.9}$$

Since we consider only linear transformations, we can readily show that

$$\frac{\partial A_j'}{\partial x'^{j}} = \frac{\partial x^k}{\partial x'^{i}} \frac{\partial x^l}{\partial x'^{j}} \frac{\partial A_k}{\partial x^l} . \tag{1.5.10}$$

It follows that the derivative of a covariant vector is a covariant tensor of the second rank. The raising and lowering of the indices on a tensor is also accomplished using a metric tensor. We now show that the Maxwell– Lorentz equations have the same form in all systems of coordinates where the metric coefficients

$$g_{00} = 1, \; g_{11} = -1, \; g_{22} = -1, \; g_{33} = -1, \; g_{ik} = 0, \; \text{if} \; i \neq k, \tag{1.5.11}$$

remain unchanged. This means that the Maxwell–Lorentz equations are invariant with respect to the group of the motion of the metric.

To see this it is simply necessary to write the Maxwell–Lorentz equations in terms of vectors and tensors of four-dimensional space-time. Before doing so, we will express the fields **E** and **H** through the scalar and vector potentials Φ and **A**

$$\mathbf{H} = \operatorname{curl} \mathbf{A}, \quad \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad} \Phi. \tag{1.5.12}$$

For **A** and Φ we will then have

$$\Box \mathbf{A} = -\frac{4\pi}{c} \mathbf{j}, \quad \Box \Phi = -4\pi\rho, \quad \frac{1}{c} \frac{\partial \Phi}{\partial t} + \operatorname{div} \mathbf{A} = 0. \tag{1.5.13}$$

From these equations follows the charge conservation law

$$\frac{\partial \beta}{\partial t} + \operatorname{div} \mathbf{j} = 0. \tag{1.5.14}$$

For this equation to be invariant under Lorentz transformations, the charge density ρ and current **j** must be components of the four-dimensional contravariant vector S^i:

$$S^i = (c\rho, \mathbf{j}). \tag{1.5.15}$$

The continuity equation then becomes

$$\frac{\partial S^i}{\partial x^i} = 0. \tag{1.5.16}$$

The charge density is thus transformed as time, and the current **j** as the vector **R**. Using the expression (1.5.15) for the current S^i, the equations for **A** and Φ, we will have

$$\Box \mathbf{A} = -\frac{4\pi}{c} \mathbf{S},$$

$$\Box \Phi = -\frac{4\pi}{c} S^0.$$

For these equations to remain unaltered under Lorentz transformations, the scalar Φ and vector **A** potentials must be components of the four-dimensional contravariant vector A^i

$$A^i = (\Phi, \mathbf{A}). \tag{1.5.17}$$

Then

$$\Box A^i = -\frac{4\pi}{c} S^i. \tag{1.5.18}$$

Since $A_i = g_{ik} A^k$, we have

$$A_i = (\Phi, -A_x, -A_y, -A_z) = (\Phi, -\mathbf{A}). \tag{1.5.19}$$

We introduce the skew-symmetric tensor

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}. \tag{1.5.20}$$

Then always

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0. \tag{1.5.21}$$

Since

$$\begin{aligned} -H_x &= F_{23}, \quad -H_y = F_{31}, \quad -H_z = F_{12}, \\ -E_x &= F_{10}, \quad -E_y = F_{20}, \quad -E_y = F_{30}, \end{aligned} \tag{1.5.22}$$

we can easily see that the above system of equations combines the pair of Maxwell's equations

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{div} \mathbf{H} = 0.$$

We then introduce the 4-vector

$$d^i = \frac{\partial F^{ik}}{\partial x^k}, \tag{1.5.23}$$

where the tensor F^{ik} is related to the components of the fields **E** and **H** by

$$\begin{aligned} -H_x &= F^{23}, \quad -H_y = F^{31}, \quad -H_z = F^{12}, \\ E_x &= F^{10}, \quad E_y = F^{20}, \quad E_z = F^{30}. \end{aligned}$$

For this system of equations to coincide with the second pair of Max-

well's equations

$$\operatorname{curl}\mathbf{H}=\frac{4\pi}{c}\,\mathbf{j}+\frac{1}{c}\,\frac{\partial\mathbf{E}}{\partial t},\qquad \operatorname{div}\mathbf{E}=4\pi\rho,$$

it is necessary to assume that

$$d^0=-\frac{4\pi}{c}\,S^0,$$

$$d^\nu=-\frac{4\pi}{c}\,S^\nu,$$

where $\nu=1,2,3$.

We have thus obtained the tensor form of the Maxwell–Lorentz system of equations in four-dimensional space-time

$$\frac{\partial F^{ik}}{\partial x^k}=-\frac{4\pi}{c}\,S^i,\quad \frac{\partial F_{ik}}{\partial x^l}+\frac{\partial F_{kl}}{\partial x^i}+\frac{\partial F_{li}}{\partial x^k}=0, \tag{1.5.24}$$

$$\frac{\partial S^i}{\partial x^i}=0,\quad F_{ik}=\frac{\partial A_k}{\partial x^i}-\frac{\partial A_i}{\partial x^k}.$$

We now write the transformation formulas for various physical quantities involved in transition from one inertial frame of reference to another in Galilean coordinates. For simplicity we will think of K'-frame as moving relative to K-system with a uniform velocity V along the X-axis. The transformation law for the covariant vector A^i is analogous to the transformation law for the coordinates of the vector $x^i = (cT, X, Y, Z)$

$$A'^0=\gamma\left(A^0-\frac{V}{c}\,A^1\right),\quad A'^1=\gamma\left(A^1-\frac{V}{c}A^0\right), \tag{1.5.25}$$

$$A'^2=A^2,\qquad A'^3=A^3.$$

Using these formulas, we derive the transformation law for charge and current:

$$\rho'=\gamma\rho\left(1-\frac{Vv_x}{c^2}\right),\quad \rho'v'_x=\gamma\rho(v_x-V), \tag{1.5.26}$$

$$\rho'v'_y=\rho v_y,\qquad \rho'v'_z=\rho v_z.$$

These relations were first obtained by Poincaré. Substituting ρ' from the first of these into the remaining ones, we will arrive at the velocity

transformation law

$$v'_x = \frac{v_x - V}{1 - \dfrac{Vv_x}{c^2}}, \quad v'_y = \gamma^{-1} \frac{v_y}{1 - \dfrac{Vv_x}{c^2}}, \quad v'_y = \gamma^{-1} \frac{v_z}{1 - \dfrac{Vv_x}{c^2}} \tag{1.5.27}$$

Since $A^i = (\Phi, \mathbf{A})$, for the scalar and vector potentials we have

$$\Phi' = \gamma\left(\Phi - \frac{V}{c} A_x\right), \quad A'_x = \gamma\left(A_x - \frac{V}{c}\Phi\right), \tag{1.5.28}$$

$$A'_y = A_y, \quad A'_z = A_z .$$

These formulas are also due to Poincaré.

The component transformation law for the skew-symmetric tensor F^{ik} coincides with the transformation formula for the tensor constructed out of the vectors B^i and C^k

$$B^i C^k - C^i B^k .$$

Since we know the formulas to transform vectors, we can readily derive formulas for the transformation of the components of the skew-symmetric tensor F^{ik}

$$F'^{12} = \gamma\left(F^{12} - \frac{V}{c} F^{02}\right), \quad F'^{02} = \gamma\left(F^{02} - \frac{V}{c} F^{12}\right).$$

Similarly,

$$F'^{13} = \gamma\left(F^{13} - \frac{V}{c} F^{03}\right), \quad F'^{03} = \gamma\left(F^{03} - \frac{V}{c} F^{13}\right). \tag{1.5.29}$$

Two other components remain the same

$$F'^{01} = F^{01}, \quad F'^{23} = F^{23}.$$

Recalling the connection between the components of F^{ik} and the components of electric and magnetic fields, we will find from these formulas the transformation law for the components of the electric field

$$E'_x = E_x, \quad E'_y = \gamma\left(E_y - \frac{V}{c} H_z\right), \tag{1.5.30}$$

$$E'_z = \gamma\left(E_z + \frac{V}{c} H_y\right)$$

and for the components of the magnetic field

$$H'_x = H_x, \quad H'_y = \gamma\left(H_y + \frac{V}{c} E_z\right), \quad H'_z = \gamma\left(H_z - \frac{V}{c} E_y\right). \tag{1.5.31}$$

These formulas were first derived by Lorentz, but he did not go so far as to establish their group character. This was done by Poincaré.

It follows from the transformation formulas for the electric and magnetic fields that if, for instance, in the K'-frame the magnetic field is zero, then in another frame of reference it will be nonzero and equal to

$$H_y = -\frac{V}{c} E_z, \tag{1.5.32}$$

$$H_z = \frac{V}{c} E_y,$$

or

$$\mathbf{H} = \frac{1}{c} [\mathbf{VE}].$$

Likewise, if electric field is zero in the frame K', then in the frame K is nonzero and equal to

$$\mathbf{E} = -\frac{1}{c} [\mathbf{VH}]. \tag{1.5.33}$$

Out of the field components we can construct two combinations invariant with respect to the Lorentz transformations

$$E^2 - H^2 = \frac{1}{2} F_{ik} F^{ik}, \quad \mathbf{EH} = \frac{1}{4} F^*_{ik} F^{ik}, \tag{1.5.34}$$

where $F^{*ik} = \epsilon^{iklm} F_{lm}$, $\epsilon^{iklm} = -\epsilon^{kilm} = \epsilon^{klim} = -\epsilon^{klmi}$, $\epsilon^{0123} = 1$ (ϵ^{iklm} is the Levi–Civita tensor). These invariants of the electromagnetic field were first discovered by Poincaré. It follows, in particular, that if in the K-system the fields $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to each other but not equal, then we can always find a frame of reference in which the field will be purely electric or purely magnetic depending on the sign of the first invariant. The fields $\mathbf{E}$ and $\mathbf{H}$ mutually perpendicular in one reference frame remain so in any other frame.

We will now obtain another expression for the four-vector of the current. Consider the invariant

$$S_i S^i = c^2 \rho^2 \left(1 - \frac{V^2}{c^2}\right). \tag{1.5.35}$$

We denote by ρ_0 the charge density in the reference frame where the charge is at rest, then

$$c^2 \rho^2 \left(1 - \frac{V^2}{c^2}\right) = c^2 \rho_0^2.$$

Hence

$$\rho = \gamma \rho_0. \tag{1.5.36}$$

We introduce the velocity vector

$$U^i = (c\gamma, \gamma \mathbf{v}), \qquad U_i U^i = c^2, \tag{1.5.37}$$

whose components are transformed as time and coordinates. This quantity was first introduced by Poincaré. The charge conservation equation can now be written as

$$\frac{\partial(\rho_0 U^i)}{\partial x^i} = 0. \tag{1.5.38}$$

We now find the transformation law for the force in transition from one inertial frame to another. The expression for the Lorentz force per unit volume in the K-frame has the form

$$\mathbf{F} = \rho \mathbf{E} + \rho \frac{1}{c} \, [\mathbf{u}, \mathbf{H}]. \tag{1.5.39}$$

In the K'-system we will then have a similar expression

$$\mathbf{F}' = \rho' \mathbf{E}' + \rho' \frac{1}{c} \, [\mathbf{u}', \mathbf{H}']. \tag{1.5.40}$$

Expressing primed variables through unprimed ones, we will obtain the transformation law for the force

$$F'_x = \gamma\left(F_x - \frac{V}{c} F\right), \quad F'_y = F_y, \quad F'_z = F_z, \tag{1.5.41}$$

$$F' = \gamma\left(F - \frac{V}{c} \, F_x\right).$$

Here

$$F = \frac{1}{c}\,(\mathbf{uF}). \tag{1.5.42}$$

These formulas were first obtained by Poincaré. We see thus that the three-dimensional scalar F and vector $\mathbf{F}$ are transformed in a manner like the components $(X^0, \mathbf{X})$. We now establish the transformation law per unit charge

$$\mathbf{f} = \mathbf{E} + \frac{1}{c}\,[\mathbf{uH}], \quad \mathbf{f} = \frac{\mathbf{E}}{\rho}. \tag{1.5.43}$$

Using (1.5.41) and (1.5.42) gives

$$f'_x = \gamma\frac{\rho}{\rho'}\left(f_x - \frac{V}{c}\,f\right), \quad f' = \gamma\frac{\rho}{\rho'}\left(f - \frac{V}{c}\,f_x\right), \tag{1.5.44}$$

$$f'_y = \frac{\rho}{\rho'}\,f_y\,, \quad f'_z = \frac{\rho}{\rho'}\,f_z\,,$$

where $f = (\mathbf{uf})/c$. From (1.5.26) we have

$$\frac{\rho}{\rho'} = \frac{\sqrt{1 - \dfrac{V^2}{c^2}}}{1 - \dfrac{u_x V}{c^2}}. \tag{1.5.45}$$

For our further arguments it is convenient to introduce the relative velocity and make use of the formula (1.3.15). This formula can easily be obtained in terms of the concept of the four-vector of velocity. We take two bodies. Suppose that the vector of velocity for each of them in some inertial frame of reference K has the components

$$U_1 = (c\gamma_1, \mathbf{V}_1\gamma_1), \quad U_2 = (c\gamma_2, \mathbf{V}_2\gamma_2).$$

Then in the K'-frame, where the first body is at rest, we have

$$U_1 = (c, 0), \quad U_2 = (c\gamma, \mathbf{V}\gamma).$$

Since the quantity

$$U_{1i}\,U_2^i$$

is an invariant, then computing it in the frames K and K', we will have

$$1 - \frac{V^2}{c^2} = \frac{(1 - V_1^2/c^2)(1 - V_2^2/c^2)}{\left(1 - \dfrac{\mathbf{V}_1\mathbf{V}_2}{c^2}\right)^2},$$

hence

$$V^2 = \frac{(\mathbf{V}_1 - \mathbf{V}_2)^2 - [\mathbf{V}_1\mathbf{V}_2]^2/c^2}{\left(1 - \dfrac{\mathbf{V}_1\mathbf{V}_2}{c^2}\right)^2}.$$

Applying here the previous form we will find for the velocity u'^2 in the K'-frame the following expression:

$$\sqrt{1 - \frac{u'^2}{c^2}} = \frac{\sqrt{1 - V^2/c^2}\,\sqrt{1 - u^2/c^2}}{1 - \dfrac{u_x V}{c^2}}. \qquad (1.5.46)$$

Using it, we will get

$$\frac{\rho}{\rho'} = \frac{\sqrt{1 - u'^2/c^2}}{\sqrt{1 - u^2/c^2}}. \qquad (1.5.47)$$

Substituting this expression into the transformation formulas (1.5.44) and introducing the notation

$$\underline{\mathfrak{R}} = \frac{f}{\sqrt{1 - u^2/c^2}}, \qquad \mathfrak{R} = \frac{f}{\sqrt{1 - u^2/c^2}}, \qquad (1.5.48)$$

we obtain

$$\mathfrak{R}'_x = \gamma\left(\mathfrak{R}_x - \frac{V}{c}\,\mathfrak{R}\right), \quad \mathfrak{R}' = \gamma\left(\mathfrak{R} - \frac{V}{c}\,\mathfrak{R}_x\right), \qquad (1.5.49)$$

$$\mathfrak{R}'_y = \mathfrak{R}_y, \qquad \mathfrak{R}'_z = \mathfrak{R}_z.$$

We see that the geometrical object $\{\mathfrak{R}, \underline{\mathfrak{R}}\}$ is transformed as $\{x^0, \mathbf{x}\}$. The quantity was introduced by Poincaré. It is generally called the four-vector of the force

$$\mathfrak{R}^i = (\mathfrak{R}, \underline{\mathfrak{R}}). \qquad (1.5.50)$$

We now write the Lorentz force in the four-dimensional form. We easily see that

$$F^i = \frac{1}{c}\, F^{ik} S_k. \qquad (1.5.51)$$

If we take into account the formulas (1.5.15) and (1.5.23), we will immediately find that

$$\mathbf{F} = \rho\mathbf{E} + \frac{\rho}{c}\,[\mathbf{uH}], \quad F^0 = \frac{1}{c}\,(\mathbf{uF}).$$

Similarly, for the four-vector of the force we have

$$\mathfrak{R}^i = \frac{1}{c} F^{ik} U_k. \qquad (1.5.52)$$

The entire system of the equations of electrodynamics, including the Lorentz force, is written in terms of vectors and tensors of four-dimensional space-time. The ideas of space-time, which have had their beginnings in electrodynamics, can now be extended to include all physical phenomena, notably mechanical ones. This agrees completely with the principle of relativity – absolute motion is impossible in nature. This suggests that the equations of Newtonian mechanics have to be changed for the new equations to be invariant under Lorentz transformations. The easiest way to accomplish this is to write them in four-dimensional form. Although we have only discussed electrodynamics so far, nevertheless some formulas we have derived are general in nature. Above all, this applies to the concept of the four-vector of velocity (1.5.37) and the four-vector of the force (1.5.48) and (1.5.49).

We now turn to the relativistic equations of mechanics.

1.6. Poincaré's Relativistic Mechanics

We will make use of the four-vector of velocity (1.5.37) and introduce the four-vector of momentum of a particle

$$p^i = mU^i, \qquad p_1 p^i = m^2 c^2 . \qquad (1.6.1)$$

Since the velocity of the particle is at all times less than c, we will define the invariant time $d\tau$

$$ds^2 = c^2 d\tau^2 = c^2 \left(1 - \frac{V^2}{c^2}\right) dt^2 . \qquad (1.6.2)$$

The derivative of the four-vector of velocity with respect to the invariant time τ will also be a four-vector. It is generally known as the four-vector of acceleration.

Taking account of the definition of the four-vector of the force (1.5.48), the relativistic equations of mechanics will be

$$m \frac{dU^i}{d\tau} = \mathfrak{R}^i, \qquad (1.6.3)$$

or, in three dimensions,

$$\frac{d}{dt}\left(\frac{m\mathbf{V}}{1-\dfrac{V^2}{c^2}}\right)=\mathbf{f}, \tag{1.6.4}$$

$$\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\dfrac{V^2}{c^2}}}\right)=(\mathbf{Vf}). \tag{1.6.5}$$

These equations were derived by Poincare, who is credited with the creation of relativistic mechanics.

The second of these can be derived from the first one, by multiplying both sides of the equations by $\mathbf{V}$. From (1.6.4) and (1.6.5), we can readily determine the momentum $\mathbf{p}$ and energy E of the particle

$$\mathbf{p}=\frac{m\mathbf{V}}{\sqrt{1-\dfrac{V^2}{c^2}}}, \qquad E=\frac{mc^2}{\sqrt{1-\dfrac{V^2}{c^2}}}. \tag{1.6.6}$$

Then

$$p^i=\left(\frac{E}{c},\ \mathbf{p}\right). \tag{1.6.7}$$

It can readily be seen that

$$\mathfrak{R}^i p_i = 0. \tag{1.6.8}$$

Expressions (1.6.6) can also be derived using the Lagrange function, if we define it as follows:

$$L=-mc^2\sqrt{1-V^2/c^2}. \tag{1.6.9}$$

Then the momentum $\mathbf{p}$ will be

$$\mathbf{p}=\frac{\partial L}{\partial \mathbf{V}}=\frac{m\mathbf{V}}{\sqrt{1-V^2/c^2}}, \tag{1.6.10}$$

and since the Hamiltonian H is

$$H=\mathbf{V}\frac{\partial L}{\partial \mathbf{V}}-L, \tag{1.6.11}$$

we have

$$E=\frac{mc^2}{\sqrt{1-V^2/c^2}} \quad \text{or} \quad E=c\sqrt{p^2+m^2c^2}. \tag{1.6.12}$$

The action function for the particle has the form

$$S = \int_{t_1}^{t_2} L\,dt, \tag{1.6.13}$$

or, considering that in Galilean coordinates

$$ds = c\,dt\sqrt{1 - \frac{V^2}{c^2}},$$

we have

$$S = -mc \int_a^b ds. \tag{1.6.14}$$

The Lagrange function (1.6.9) was first formulated by Poincaré. Here the integration is between two fixed points along the world line. In an arbitrary permissible frame of reference the interval has the form

$$ds^2 = g_{ik}\,dx^i dx^k, \tag{1.6.15}$$

and so the Lagrange function for the particle is

$$L = -mc^2\sqrt{g_{00} + \frac{1}{c}\,2g_{0\alpha}\dot{x}^\alpha + \frac{1}{c^2}\,g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}. \tag{1.6.16}$$

The generalized momenta will thus be

$$\frac{\partial L}{\partial \dot{x}^\alpha} = (mc^2)^2\,\frac{\frac{1}{c}\,g_{0\alpha} + \frac{1}{c^2}\,g_{\alpha\beta}\dot{x}^\beta}{L}. \tag{1.6.17}$$

The Hamilton function is computed in the conventional manner:

$$H = \frac{\partial L}{\partial \dot{x}^\alpha}\,\dot{x}^\alpha - L. \tag{1.6.18}$$

Considering that

$$\dot{x}^\alpha\,\frac{\partial L}{\partial \dot{x}^\alpha} = L - (mc^2)^2\,\frac{g_{00} + \frac{1}{c}\,g_{0\beta}\dot{x}^\beta}{L},$$

we find that

$$H = -(mc^2)^2 \frac{g_{00} + \frac{1}{c} g_{0\beta}\dot{x}^\beta}{L} . \qquad (1.6.19)$$

We now introduce the four-vector of the momentum

$$p_i = mcg_{ik} \frac{dx^k}{ds}$$

here

$$p_0 = \frac{H}{c} \qquad (1.6.20)$$

or

$$p^i = mc \frac{dx^i}{ds}. \qquad (1.6.21)$$

Since

$$g_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} = 1, \qquad (1.6.22)$$

we have

$$g_{ik}p^i p^k = m^2 c^2 . \qquad (1.6.23)$$

Similarly

$$g^{ik} p_i p_k = m^2 c^2 . \qquad (1.6.24)$$

We now express the momenta through partial derivatives of the action function S with respect to the coordinates and time

$$p_i = \frac{\partial S}{\partial x^i} . \qquad (1.6.25)$$

Substituting these expressions into (1.6.24), we will obtain the equation for the geodesic in the Hamilton–Jacobi form

$$g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} = m^2 c^2 . \qquad (1.6.26)$$

We see that the relativistic equations for the motion of a particle, just like the equations for the front of an electromagnetic wave, incorporate some ideas of the structure of space-time. In Galilean coordinates,

where the metric tensor is

$$g_{00} = 1, \quad g_{11} = -1, \quad g_{22} = -1, \quad g_{33} = -1, \quad g_{ik} = 0, \quad \text{if} \; i \neq k,$$

the equation for the geodesic will be

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2 - \left(\frac{\partial S}{\partial z}\right)^2 = m^2 c^2. \qquad (1.6.27)$$

If instead of S we introduce the action in the following manner:

$$S \to S - mc^2 t,$$

then, after the change, in the limit as $c \to \infty$, we will obtain the Hamilton – Jacobi equation (1.1.14) in classical mechanics.

1.7. Stationary Action Principle in Electrodynamics

Many equations of theoretical physics are derived from the extremum condition for the functional that is called action. For electrodynamics too we will have to set up action in such a manner that its variation in the fields would yield Maxwell – Lorentz equations. The action is formulated using scalar functions of the field and current.

We form the action that contains the electromagnetic field and its interaction with the charges and currents. We introduce the tensor of the electromagnetic field

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}, \qquad (1.7.1)$$

which by virtue of its construction satisfies Maxwell's equation

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0. \qquad (1.7.2)$$

Potentials A_i will be varied later in the book to obtain other field equations. For our further discussion we will only need the two invariants

$$A_i S^i \quad \text{and} \quad F_{ik} F^{ik}. \qquad (1.7.3)$$

Here S^i is the four-dimensional vector of the current.

As we will see later, these scalar functions are quite sufficient to construct an action whose variation will lead to the second pair of Maxwell's equations. The required action will be

$$S = \frac{1}{c} \int \mathfrak{L} \, d\Omega,$$

where $\mathfrak{L}$ is the density of the Lagrange function

$$\mathfrak{L} = -\frac{1}{c} A_i S^i - \frac{1}{16\pi} F_{ik}F^{ik}, \text{ a } \quad d\Omega = dx^0 dx^1 dx^2 dx^3. \tag{1.7.4}$$

In deriving the field equation we will in the action function vary only field potentials, while the sources of the field S^i are considered to be specified. Then

$$\delta S = -\frac{1}{c} \int \left[\frac{1}{c} S^i \delta A_i + \frac{1}{8\pi} F^{ik} \delta F_{ik}\right] d\Omega.$$

Expressing F_{ik} through the potentials A_i, we obtain

$$\delta S = -\frac{1}{c} \int \left[\frac{1}{c} S^i \delta A_i - \frac{1}{4\pi} F^{ik} \frac{\partial}{\partial x^k} \delta A_i\right] d\Omega = 0.$$

Integrating in the second term by parts and considering that potential variations at the initial and final moments of time are zero, and the field vanishes at infinity, we will find that

$$\delta S = -\frac{1}{c} \int \left[\frac{1}{c} S^i + \frac{1}{4\pi} \frac{\partial F^{ik}}{\partial x^k}\right] \delta A_i d\Omega = 0.$$

Since the variations δA_i are arbitrary, we have

$$\frac{\partial F^{ik}}{\partial x^{ik}} = -\frac{4\pi}{c} S^i. \tag{1.7.5}$$

We see thus that our choice of the action is justified, since we have really obtained the second pair of Maxwell's equations. It should be borne in mind, however, that the choice of the density of the Lagrange function in the action function is not unique, since we can readily make sure that the addition to the density of the Lagrange function of an additional term of the type of the four-dimensional divergence of the vector does affect the form of the field equations. In particular, we note that Maxwell's equations (1.5.24) are invariant under gauge transformations of the potentials

$$A' = A_i + \frac{\partial f}{\partial x^i}, \tag{1.7.6}$$

where f is an arbitrary function.

The Lagrange density $\mathfrak{L}$ we have just constructed is not invariant; it changes under transformations by the four-dimensional divergence of the vector. We now find the equations of motion of charge particles

in an electromagnetic field. To obtain them we will have to form an action that contains a part referring to particles, along with the already-known part, which includes the interaction of the field with the particles.

Since for the particles

$$\rho = e\delta(\mathbf{r} - \mathbf{r}_0), \qquad \mathbf{j} = e\mathbf{v}\delta(\mathbf{r} - \mathbf{r}_0), \tag{1.7.7}$$

we have

$$-\frac{1}{c^2}\int S^i A_i d\Omega = -\frac{e}{c}\int A_i dx^i, \tag{1.7.8}$$

the action for the particles in the electromagnetic field will be

$$S = -mc\int ds - \frac{e}{c}\int A_i dx^i. \tag{1.7.9}$$

Taking the variations in the particle's coordinates gives

$$\delta S = -\int\left(mc\,\frac{dx_i}{ds}\,d\delta x^i + \frac{e}{c}A_i d\delta x^i + \frac{e}{c}\,\delta A_k dx^k\right) = 0.$$

Integrating in the first two terms by parts and assuming the variations of the coordinates at the ends to be zero, we get

$$\delta S = \int\left(mdu_i\delta x^i + \frac{e}{c}\,dA_i\delta x^i - \frac{e}{c}\,\delta A_k\,dx^k\right) = 0.$$

Substituting into this expression the obvious relations

$$dA_i = \frac{\partial A_i}{\partial x^k}dx^k, \qquad \delta A_k = \frac{\partial A_k}{\partial x^i}\,\delta x^i,$$

we obtain

$$\int\left[m\,\frac{du_i}{ds} + \frac{e}{c^2}\,\frac{\partial A_i}{\partial x^k}\,u^k - \frac{e}{c^2}\,\frac{\partial A_k}{\partial x^i}\,u^k\right]\delta x^i ds = 0.$$

The variations δx^i being arbitrary, we obtain from this the equation of motion for the charge in the form

$$mc\,\frac{du_i}{ds} = \frac{e}{c}\,F_{ik}u^k \tag{1.7.10}$$

or

$$mc\,\frac{du^i}{ds} = \frac{e}{c}\,F^{ik}u_k. \tag{1.7.11}$$

1.8. Electrodynamics in Arbitrary Coordinates

Before presenting Maxwell's equations in arbitrary coordinates, we notice that the conventional derivative of a vector in these variables is no longer a tensor of the second rank, as was the case earlier, when we have only dealt with the coordinates for which all the metric coefficients g_{ik} of the pseudo-Euclidean space-time are constant. For our further discussion we will have to introduce covariant differentiations. We will define the covariant differentiation of a vector as a differentiation that yields a tensor of the second rank.

The covariant derivative of a covariant vector will then be

$$\nabla_i A_k = \frac{\partial A_k}{\partial x^i} - \Gamma^s_{ki} A_s, \tag{1.8.1}$$

and of a contravariant vector

$$\nabla_i A^k = \frac{\partial A^k}{\partial x^i} + \Gamma^k_{is} A^s, \tag{1.8.2}$$

where Γ^s_{ki} are Christoffel's symbols, which are given by expression (1.8.5).

We define the covariant derivative of a tensor of the second rank for a covariant tensor as

$$\nabla_i T_{kl} = \frac{\partial T_{kl}}{\partial x^i} - \Gamma^s_{ki} T_{sl} - \Gamma^s_{li} T_{ks} \tag{1.8.3}$$

and for a contravariant tensor

$$\nabla_i T^{kl} = \frac{\partial T^{kl}}{\partial x^i} + \Gamma^k_{si} T^{sl} + \Gamma^l_{si} T^{ks}. \tag{1.8.4}$$

These formulas for derivatives are easily generalized to tensors of any rank.

Christoffel's symbols are expressed in terms of metric coefficients as

$$\Gamma^k_{lm} = g^{kn} \Gamma_{n;lm}, \quad \Gamma_{n;lm} = \frac{1}{2} \left(\frac{\partial g_{nm}}{\partial x^l} + \frac{\partial g_{nl}}{\partial x^m} - \frac{\partial g_{lm}}{\partial x^n} \right). \tag{1.8.5}$$

We will define covariant differentiation as the operation

$$\nabla^i = g^{ik} \nabla_k. \tag{1.8.6}$$

We now compute the covariant divergence of a vector

$$\nabla_k A^k = \frac{\partial A^k}{\partial x^k} + \Gamma^k_{ki} A^i. \tag{1.8.7}$$

Here

$$\Gamma^k_{ki} = g^{ml} \Gamma_{m;li} = \frac{1}{2} g^{ml} \frac{\partial g_{ml}}{\partial x^i}.$$

Considering that

$$g^{ml} \frac{\partial g_{ml}}{\partial x^i} = \frac{1}{g} \frac{\partial g}{\partial x^i}, \tag{1.8.8}$$

where g is a determinant composed of g_{ik}, we find

$$\Gamma^k_{ki} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \sqrt{-g}. \tag{1.8.9}$$

Using this expression, we obtain

$$\nabla_k A^k = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} A^k)}{\partial x^k}. \tag{1.8.10}$$

We now turn to Maxwell's equations in curvilinear coordinates of space-time and introduce the tensor of electromagnetic field

$$F_{ik} = \nabla_k A_i - \nabla_i A_k = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}. \tag{1.8.11}$$

We see that the terms containing Christoffel's symbols cancel out and do not thus enter into the expression for F_{ik} through the potentials. Hence the first pair of Maxwell's equations can be written in the conventional form

$$\nabla_i F_{kl} + \nabla_k F_{li} + \nabla_l F_{ik} = \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} + \frac{\partial F_{ik}}{\partial x^l} = 0. \tag{1.8.12}$$

The other pair of Maxwell's equations has the form

$$\nabla_k F^{ik} = -\frac{4\pi}{c} \rho_0 u^i. \tag{1.8.13}$$

The equations of the motion of charged particles can be deduced by

generalizing the equation (1.7.10)

$$\mu_0 u^k \nabla_k u_i = \frac{\rho_0}{c} F_{il} u^l, \tag{1.8.14}$$

where μ_0 is the density of the rest mass.

The tensor F^{ik} being skew-symmetric, we find from (1.8.13) that the vector $\rho_0 u^i$ obeys

$$\nabla_k (\rho_0 u^k) = 0. \tag{1.8.15}$$

Likewise, the conservation equation is obeyed by the vector $\mu_0 u^k$

$$\nabla_k (\mu_0 u^k) = 0. \tag{1.8.16}$$

The mass density $\mu = \sqrt{-g}\, \mu_0 u^0$ is said to be conserved, since the integral of μ over the volume is independent of time by virtue of (1.8.16).

We now show that the covariant Maxwell equations can be obtained from the variation of the action over variables of the field and particles, if the action is written for arbitrary coordinates. The transition in the action from Galilean coordinates, in which the metric tensor is diagonal (1, −1, −1, −1), the curvilinear coordinates, in which the metric coefficients are functions of coordinates, is readily performed using a simple change of variables in the integrant and an elementary tensor analysis. After this procedure we obtain the action function in arbitrary coordinates

$$S = -\frac{1}{2} \int \sqrt{-g}\, d\Omega \left[\mu_0 c^2 + \frac{1}{c} \rho_0 u^i A_i + \frac{1}{16\pi} F_{ik} F^{ik} \right]. \tag{1.8.17}$$

Note that under any coordinate transformations the quantity

$$\sqrt{-g}\, dx^0 dx^1 dx^2 dx^3 \tag{1.8.18}$$

remains an invariant.

We now proceed to derive, following Fock's procedure, the covariant Maxwell equations and the equations of motion of matter on the basis of the principle of stationary action. To this end, we will have to compute the variations of the quantities u^α, ρ_0 and μ_0 for arbitrary variations of the trajectories of matter particles. We denote by a_1, a_2, a_3 the initial coordinates of a particle, and by p the parameter characterizing time. We then express the coordinates of the particle in terms of the new variables

$$x^k = f^k (p, a_1, a_2, a_3), \quad k = 0, 1, 2, 3. \tag{1.8.19}$$

These functions describe the motion of the particle with constant a_1,

a_2, a_3 and variable p. Later in the book, when deriving the equations of motion of matter we will vary with respect to these functions:

$$\delta x^i = \delta f^i(p, a_1, a_2, a_3). \tag{1.8.20}$$

In terms of the new variables, the components of the four-dimensional velocity will have the form

$$u^l = c \frac{\dfrac{\partial f^l}{\partial p}}{\sqrt{g_{ik} \dfrac{\partial f^i}{\partial p} \dfrac{\partial f^k}{\partial p}}}. \tag{1.8.21}$$

It is easily seen that

$$u_k u^k = g_{kl} u^k u^l = c^2. \tag{1.8.22}$$

We introduce the definition of the variations. We define

$$\delta_c u^i = u'^i(x + \delta x) - u^i(x). \tag{1.8.23}$$

The variation $\delta_c u^i$ is no vector. We introduce the variation $\delta u^i(x)$

$$\delta u^i(x) = u'^i(x) - u^i(x). \tag{1.8.24}$$

It is easily seen that

$$\delta u^i(x) = \delta_c u^i - \frac{\partial u^i}{\partial x^k} \delta x^k. \tag{1.8.25}$$

We now compute the variation δ_c of the function

$$\Phi = \sqrt{g_{ik} \frac{\partial f^i}{\partial p} \frac{\partial f^k}{\partial p}}. \tag{1.8.26}$$

We find that

$$\delta_c \Phi = \frac{1}{c^2} \Phi u^i u^k \left(\frac{1}{2} \frac{\partial g_{ik}}{\partial x^l} \delta x^l + g_{li} \frac{\partial \delta x^l}{\partial x^k} \right). \tag{1.8.27}$$

Using the formula for the covariant derivative

$$\nabla_i \delta x^k = \frac{\partial \delta x^k}{\partial x^i} + \Gamma^k_{il} \delta x^l, \tag{1.8.28}$$

we get

$$\delta_c \left(\sqrt{g_{ik} \frac{\partial f^i}{\partial p} \frac{\partial f^k}{\partial p}} \right) = \frac{1}{c^2} \sqrt{g_{ik} \frac{\partial f^i}{\partial p} \frac{\partial f^k}{\partial p}} \, u_l u^m \nabla_m \delta x^l. \tag{1.8.29}$$

Considering that

$$\frac{\partial \delta x^i}{\partial p} = \frac{\partial f^k}{\partial p} \frac{\partial \delta x^i}{\partial x^k}, \tag{1.8.30}$$

we obtain from (1.8.29)

$$\delta_c u^i = u^k \frac{\partial \delta x^i}{\partial x^k} - \frac{1}{c^2} u^i u_n u^k \nabla_k \delta x^n. \tag{1.8.31}$$

By (1.8.25), we will have

$$\delta u^i(x) = u^k \frac{\partial \delta x^i}{\partial x^k} - \delta x^k \frac{\partial u^i}{\partial x^k} - \frac{1}{c^2} u^i u_n u^k \nabla_k \delta x^n, \tag{1.8.32}$$

or, in covariant form

$$\delta u^i(x) = u^n \nabla_n \delta x^i - \delta x^n \nabla_n u^i - \frac{1}{c^2} u^i u_k u^n \nabla_n \delta x^k. \tag{1.8.33}$$

We see that the variation δu^i is a vector. Using the continuity equations

$$\nabla_i(\rho_0 u^i) = 0, \quad \nabla_i(\mu_0 u^i) = 0, \tag{1.8.34}$$

we will find the variations $\delta \rho_0$ and $\delta \mu_0$. By the definition of the variation δ we have

$$\nabla_i \delta(\rho_0 u^i) = 0. \tag{1.8.35}$$

Taking into account the expressions (1.8.33), we find that

$$\delta(\rho_0 u^i) = u^i \left[\delta \rho_0 + \nabla_n(\rho_n \delta x^n) - \frac{\rho_0}{c^2} u_k u^n \nabla_n \delta x^k \right]$$
$$+ \nabla_n(\rho_0 u^n \delta x^i - \rho_0 u^i \delta x^n). \tag{1.8.36}$$

Substituting this expression into (1.8.35) gives

$$\nabla_i u^i \left[\delta \rho_0 + \nabla_n(\rho_0 \delta x^n) - \frac{1}{c^2} \rho_0 u_k u^n \nabla_n \delta x^k \right] = 0, \tag{1.8.37}$$

hence

$$\delta \rho_0 = -\nabla_n(\rho_0 \delta x^n) + \frac{1}{c^2} \rho_0 u_k u^n \nabla_n \delta x^k. \tag{1.8.38}$$

Similarly,

$$\delta \mu_0 = -\nabla_n(\mu_0 \delta x^n) + \frac{1}{c^2} \mu_0 u_k u^n \nabla_n \delta x^k. \tag{1.8.39}$$

By (1.8.38) the variation of the current is

$$\delta(\rho_0 u^i) = \nabla_k(\rho_0 u^k \delta x^i - \rho_0 u^i \delta x^k). \tag{1.8.40}$$

Using the relation (1.8.10) we can write it in the form

$$\delta(\rho_0 u^i) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} [\sqrt{-g}(\rho_0 u^k \delta x^i - \rho_0 u^i \delta x^k)]. \tag{1.8.41}$$

Using the variations for $\delta\rho_0$, $\delta\mu_0$, $\delta(\rho_0 u^i)$, we can readily find the generally covariant field equations (1.8.13) and the equations of motion of matter (1.8.14). For this purpose, we will find the variation of the action

$$S = -\frac{1}{c}\int\sqrt{-g}\,d\Omega\left[\mu_0 c^2 + \frac{1}{c}\rho_0 u^i A_i + \frac{1}{16\pi} F_{ik}F^{ik}\right], \tag{1.8.42}$$

and will assume the variations δx^i, δA_k on the boundary to be zero. We now proceed to compute the variation

$$\delta S = -\delta\int\sqrt{-g}\,d\Omega\left[\mu_0 c + \frac{1}{c^2}\rho_0 u^i A_i + \frac{1}{16\pi c} F_{jk}F^{jk}\right] = 0\,. \tag{1.8.43}$$

Using (1.8.39), we find the variation of the first term in (1.8.43)

$$-\delta\int\sqrt{-g}\,d\Omega\mu_0 c = \frac{1}{c}\int \mu_0 \delta x^j (u^k \nabla_k u_j)\sqrt{-g}\,d\Omega. \tag{1.8.44}$$

From (1.8.41), the variation of the second term in (1.8.43) will be

$$-\frac{1}{c^2}\delta\int\rho_0 u^i A_i \sqrt{-g}\,d\Omega = -\frac{1}{c^2}\int\rho_0 u^i(\delta A_i + F_{ki}\delta x^k)\sqrt{-g}\,d\Omega. \tag{1.8.45}$$

We now find the variation of the third term:

$$-\frac{1}{16\pi c}\delta\int F_{ik}F^{ik}\sqrt{-g}\,d\Omega$$

$$= -\frac{1}{8\pi c}\int F^{ik}\left(\frac{\partial\delta A_k}{\partial x^i} - \frac{\partial\delta A_i}{\partial x^k}\right)\sqrt{-g}\,d\Omega. \tag{1.8.46}$$

Integrating in (1.8.46) by parts and taking account of the fact that

the tensor F^{ik} is skew-symmetric, we obtain

$$-\frac{1}{16\pi c}\,\delta\int F_{ik}F^{ik}\sqrt{-g}\,d\Omega = -\frac{1}{4\pi c}\int \nabla_j F^{ij}\,\delta A_i\sqrt{-g}\,d\Omega. \tag{1.8.47}$$

Combining (8.44), (8.45) and (8.47), we find

$$\delta S = \frac{1}{c}\int\sqrt{-g}\,d\Omega\left[\delta x^k\left(\mu_0 u^i\nabla_i u_k - \frac{1}{c}\rho_0 F_{ki}u^i\right)\right.$$
$$\left. - \left(\frac{1}{4\pi}\nabla_i F^{ki} + \frac{1}{c}\rho_0 u^k\right)\delta A_k\right] = 0. \tag{1.8.48}$$

Since δx^i and δA_k are arbitrary, we find from this

$$\mu_0 u^i\nabla_i u^k = \frac{1}{c}\rho_0 F^{kj}u_j,$$

$$\nabla_i F^{ki} = -\frac{4\pi}{c}\rho_0 u^k.$$

Thus, from the integral of the action written in arbitrary coordinates on pseudo-Euclidean space-time we have automatically obtained the generally covariant field equations and the equations of motion of matter. Notice that since in Galilean coordinates we proceeded from the diagonal metric tensor (1, −1, −1, −1) our derivation is purely mathematical in character. Although obtaining the generally covariant form of equations is a mathematical procedure, it is however exceedingly useful for descriptions of physical phenomena.

If previously the equations only allowed descriptions of electromagnetic phenomena in inertial reference frames (Galilean coordinates), then the generally covariant form of the equations enables electromagnetic phenomena to be described in noninertial (accelerated) frames. The special theory of relativity thus describes things both in inertial and in noninertial frames of reference. To be sure, these phenomena proceed in an essentially different manner in noninertial frames; however, in this case, too, we can always indicate an infinite collection of other noninertial frames, in which phenomena occur as in the original noninertial frame. And so the widely held opinion that the special theory of relativity is unapplicable to accelerated reference frames [8, 9, 11, 19, 64] is erroneous. We will look at this issue in more detail later in the book.

1.9. Equations of Motion and Conservation Laws in Classical Field Theory

We have already seen that the Lagrangian approach enables us to construct all of Maxwell's equations. This approach has a clear-cut general covariant nature. It allows the equations of motion of the field and the conservation laws to be obtained in the general form without explicit specification of the density of the Lagrange function. In this approach each physical field is described by one or multi-component function of coordinates and time, called the field function (or field variable). The field variables are quantities that are transformed according to one of the representations of the Lorentz group, e.g., scalar, spinor, vector, and even tensor representation. Along with the field variables an important role is played by the metric tensor of space-time, which defines the natural geometry for the physical field, and also the choice of one or another of coordinate systems for the physical processes to be described. The choice of the coordinate system is at the same time the choice of the frame of reference. Of course, not any choice of a coordinate system changes the reference frame. In a given frame any transformations

$$\begin{aligned} x'^0 &= f^0(x^0, x^1, x^2, x^3), \\ x'^\alpha &= f^\alpha(x^1, x^2, x^3), \quad \alpha = 1, 2, 3 \end{aligned} \tag{1.9.1}$$

always leave us in the same reference frame. Any other choice of a coordinate system is bound to bring us to another reference frame. Coordinate systems are selected from the class of the permissible coordinates in which

$$g_{00} > 0, \quad g_{\alpha\beta} dx^\alpha dx^\beta < 0. \tag{1.9.2}$$

The Lagrange formalism is derived from the action function. The expression for the action function is normally written as

$$S = \frac{1}{c} \int_\Omega \mathfrak{L}(x^0, x^1, x^2, x^3) dx^0 dx^1 dx^2 dx^3, \tag{1.9.3}$$

where the integration is over some arbitrary four-dimensional region of space-time. Since the action must be generally covariant, the density of the Lagrange function is the density of the weight scalar +1. The density of the weight scalar +1 is the product of a scalar function by the quantity $\sqrt{-g}$. The choice of the Lagrangian density is based on a number of requirements, realness and covariance among them.

The realness of the Lagrangian density guarantees that the dynamic characteristics of a physical field, e.g., of energy and momentum of the field, will be covariant. In this theory the density of the Lagrangian is

a scalar density built out of the discussed fields φ_A, the metric tensor and the partial derivative with respect to the coordinates

$$\mathfrak{L} = \mathfrak{L}\,(\varphi_A\,,\ \partial_n \varphi_A\,, \ldots,\ g_{ik},\ \partial_n g_{ik}\ \ldots). \tag{1.9.4}$$

For the sake of simplicity, we will assume that the system under consideration consists of real scalar and vector fields. Suppose also that the Lagrangian of the system of fields contains no derivatives of order higher than one. This limitation has the result that our field equations will all be second-order equations

$$\mathfrak{L} = \mathfrak{L}\,(\varphi,\ \partial_n\varphi,\ A_i,\ \partial_n A_i,\ g_{ik},\ \partial_n g_{ik},\ \ g^{ik},\ \partial_n g^{ik}). \tag{1.9.5}$$

We will find the field equations from the condition that the functional variation of the action is zero, i.e.,

$$\delta S = \frac{1}{c} \int_\Omega d\Omega \delta \mathfrak{L}\,(\varphi,\ \partial_n\varphi,\ A_i,\ \partial_n A_i,\ g_{ik},\ \partial_n g_{ik},\ g^{ik},\ \partial_n g^{ik}) = 0. \tag{1.9.6}$$

The variation $\delta\,\mathfrak{L}$ is

$$\delta\,\mathfrak{L} = \frac{\partial \mathfrak{L}}{\partial\varphi}\,\delta\varphi + \frac{\partial\,\mathfrak{L}}{\partial(\partial_n\varphi)}\,\delta\,\partial_n\varphi + \frac{\partial\mathfrak{L}}{\partial A_i}\,\delta A_i + \frac{\partial\,\mathfrak{L}}{\partial(\partial_n A_i)}\,\delta\,\partial_n A_i \tag{1.9.7}$$

or

$$\delta\,\mathfrak{L} = \frac{\delta\,\mathfrak{L}}{\delta\,\varphi}\,\delta\varphi + \frac{\delta\,\mathfrak{L}}{\delta A_i}\,\delta A_i + \partial_n \left[\frac{\partial\mathfrak{L}}{\partial(\partial_n\varphi)}\,\delta\varphi + \frac{\partial\mathfrak{L}}{\partial(\partial_n A_i)}\,\delta A_i\right] = 0. \tag{1.9.8}$$

We have here denoted in Eiler variational derivative by

$$\frac{\delta\mathfrak{L}}{\partial\varphi} = \frac{\partial\mathfrak{L}}{\partial\varphi} - \partial_n \left(\frac{\partial\mathfrak{L}}{\partial(\partial_n\varphi)}\right). \tag{1.9.9}$$

In deriving (1.9.8) we have taken into consideration that

$$\delta\,\partial_n A_i = \partial_n\,\delta A_i. \tag{1.9.10}$$

Substituting (1.9.8) into (1.9.6) and using the Gauss theorem, we obtain

$$\delta S = \frac{1}{c} \int_\Omega d^4x \left(\frac{\delta\,\mathfrak{L}}{\delta\,\varphi}\,\delta\,\varphi + \frac{\delta\,\mathfrak{L}}{\delta\,A_i}\,\delta\,A_i\right)$$
$$+ \frac{1}{c} \int ds_n \left[\frac{\partial\,\mathfrak{L}}{\partial\,(\partial_n\varphi)}\,\delta\,\varphi + \frac{\partial\,\mathfrak{L}}{\partial\,(\partial_n A_i)}\,\delta\,A\right] = 0.$$

Since the variation of the fields on the boundary is zero, we have

$$\delta S = \frac{1}{c} \int_{\Omega} d^4 x \left(\frac{\delta \mathfrak{L}}{\delta \varphi} \delta \varphi + \frac{\delta \mathfrak{L}}{\delta A_i} \delta A_i \right) = 0. \tag{1.9.11}$$

Since the variations $\delta\varphi$ and δA_i are independent and arbitrary, by the main lemma of calculus of variations we derive equations for the fields

$$\frac{\delta \mathfrak{L}}{\delta \varphi} = 0, \quad \frac{\delta \mathfrak{L}}{\delta A_i} = 0. \tag{1.9.12}$$

In addition on the field equations the Lagrange technique allows to obtain differential conservation laws. It is customary to differentiate two types of the differential laws of conservation: weak and strong.

Strong conservation is the differential relation that holds since the action is invariant under coordinate transformations. Weak conservation laws follow from the strong ones if in the latter we use the field equations (1.9.12).

It should be emphasized that the differential conservation laws in the general case do not establish conservation of anything, locally or globally. For our case, the action has the form

$$S = \frac{1}{c} \int_{\Omega} d^4 x \, \mathfrak{L}(\varphi, \partial_n \varphi, A_i, \partial_n A_i, g_{ik}, \partial_n g_{ik}, g^{ik}, \partial_n g^{ik}). \tag{1.9.13}$$

We will effect the infinitesimal transformation of the coordinate system

$$x'^i + \delta x^i, \tag{1.9.14}$$

where δx^i is an infinitesimal four-vector.

Since the action is a scalar, it remains unchanged under this transformation, and hence

$$\delta_c S = \frac{1}{c} \int_{\Omega'} d^4 x' \mathfrak{L}'(x') - \frac{1}{c} \int_{\Omega} d^4 x \, \mathfrak{L}(x) = 0, \tag{1.9.15}$$

where

$$\mathfrak{L}'(x') = \mathfrak{L}(\varphi'(x'), \; \partial'_n \varphi'(x'), \; A'_i(x'), \; \partial'_n A'_i(x),$$

$$g'_{ik}(x'), \; \partial'_n g'_{ik}(x'), \; g'^{ik}(x'), \; \partial'_n g'^{ik}(x')).$$

The first term in (1.9.15) can be written as

$$\int_{\Omega'} d^4 x' \mathfrak{L}'(x') = \int_{\Omega} J d^4 x \, \mathfrak{L}'(x'), \tag{1.9.16}$$

where the Jacobian J is

$$J = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} = \det \left\| \frac{\partial x'^i}{\partial x^k} \right\|. \tag{1.9.17}$$

Under the transformation (1.9.14) the Jacobian becomes

$$J = 1 + \partial_i \delta x^i. \tag{1.9.18}$$

Expanding $\mathfrak{L}'(x')$ into a Taylor series gives

$$\mathfrak{L}'(x') = \mathfrak{L}'(x) + \delta x^i \frac{\partial \mathfrak{L}}{\partial x^i} + \dots \tag{1.9.19}$$

From (1.9.16), (1.9.18) and (1.9.19), we can rewrite (1.9.15) as

$$\delta_c S = \frac{1}{c} \int_\Omega d^4x \left[\delta_L \mathfrak{L}(x) + \frac{\partial}{\partial x^i} (\delta x^i \mathfrak{L}(x)) \right]. \tag{1.9.20}$$

Here we have introduced the notation

$$\delta_1 \mathfrak{L}(x) = \mathfrak{L}'(x) - \mathfrak{L}(x). \tag{1.9.21}$$

This variation is generally known as the Lie variation. It is commutative with partial differentiation, i.e.,

$$\delta_L \partial_l = \partial_l \delta_L. \tag{1.9.22}$$

The Lie variation of the density of the Lagrange function is

$$\begin{aligned} \delta_1 \mathfrak{L} = {} & \frac{\partial \mathfrak{L}}{\partial \varphi} \delta_L \varphi + \frac{\partial \mathfrak{L}}{\partial(\partial_n \varphi)} \delta_L \partial_n \varphi + \frac{\partial \mathfrak{L}}{\partial A_i} \delta_L A_i \\ & + \frac{\partial \mathfrak{L}}{\partial(\partial_n A_i)} \delta_L \partial_n A_i + \frac{\partial \mathfrak{L}}{\partial g_{ik}} \delta_L g_{ik} + \frac{\partial \mathfrak{L}}{\partial(\partial_n g_{ik})} \delta_L \partial_n g_{ik} \\ & + \frac{\delta \mathfrak{L}}{\partial g^{lm}} \delta_L g^{lm} + \frac{\partial \mathfrak{L}}{\partial(\partial_n g^{lm})} \delta_L \partial_n g^{lm}. \end{aligned} \tag{1.9.23}$$

The Lie variation of the contravariant components g^{lm} is dependent on the variation of the covariant components g_{lm}. Since

$$g_{ik} g^{kl} = \delta_i^l,$$

we readily find

$$\delta_L g^{ml} = -g^{il} g^{km} \delta_L g_{ik}. \tag{1.9.24}$$

After elementary rearrangements, we find

$$\delta_c S = \frac{1}{c} \int_\Omega d^4x \left[\frac{\delta \mathfrak{L}}{\delta \varphi} \delta_L \varphi + \frac{\delta \mathfrak{L}}{\delta A_i} \delta_L A_i \right.$$

$$\left. + \left(\frac{\delta \mathfrak{L}}{\delta g_{ik}} - g^{il} g^{km} \frac{\delta \mathfrak{L}}{\delta g^{lm}} \right) \delta_L g_{ik} + D_n J^n \right] = 0, \tag{1.9.25}$$

where

$$J^n = \mathfrak{L}\, \delta x^n + \frac{\partial \mathfrak{L}}{\partial (\partial_n \varphi)} \delta_L \varphi + \frac{\partial \mathfrak{L}}{\partial (\partial_n A_i)} \delta_L A_i$$

$$+ \frac{\partial \mathfrak{L}}{\partial (\partial_n g_{ik})} \delta_L g_{ik} + \frac{\partial \mathfrak{L}}{\partial (\partial_n g^{lm})} \delta_L g^{lm} \tag{1.9.26}$$

Since J^n is the four-vector of weight +1, we have

$$\partial_n J^n = D_n J^n, \tag{1.9.27}$$

where D_n is a covariant derivative in pseudo-Euclidean space-time.

It is to be noted that the Lie variations $\delta_L \varphi$ and $\delta_L A_i$ that enter into the expression (1.9.25), unlike differential variations, are not independent of each other, since they come from the coordinate transformation (1.9.14). They can all be expressed through the four components of δx^m. The variation of the metric tensor $\delta_L g_{ik}$ is also generated by the coordinate transformation and can also be expressed through the four-vector δx^m. Since a coordinate transformation can never change the nature of geometry, our entire treatment holds good both for a pseudo-Euclidean geometry and for a Riemannian geometry, but for the sake of simplicity we only discuss the pseudo-Euclidean case.

We now find the Lie variation with respect to the field variables due to the coordinate transformation. According to the laws governing the transformations of a covariant vector

$$A'_i(x') = A_k(x) \frac{\partial x^k}{\partial x'^i} \tag{1.9.28}$$

we have

$$A'_i(x + \delta x) = A_i(x) - A_k(x) \frac{\partial \delta x^k}{\partial x^i},$$

hence

$$\delta_L A_i(x) = -\delta x^k \frac{\partial A_i}{\partial x^k} - A_k \frac{\partial \delta x^k}{\partial x^i}, \tag{1.9.29}$$

or, in covariant form,

$$\delta_L A_i(x) = -\delta x^k D_k A_i - A_k D_i \delta x^k. \tag{1.9.30}$$

Since under coordinate transformation a scalar quantity is transformed by the law

$$\varphi'(x') = \varphi(x), \tag{1.9.31}$$

we have

$$\delta_L \varphi = -\delta x^k D_k \varphi. \tag{1.9.32}$$

We now find the Lie variation of the metric tensor g_{ik}. By the law of transformation of a tensor

$$g'_{ik}(x') = \frac{\partial x^m}{\partial x'^i} \frac{\partial x^l}{\partial x'^k} g_{lm}(x)$$

we will get

$$g'_{ik}(x + \delta x) = g_{ik} - g_{il}\partial_k \delta x^l - g_{kl}\partial_i \delta x^l, \tag{1.9.33}$$

hence

$$\delta_L g_{ik} = -g_{il}\partial_k \delta x^l - g_{kl}\partial_i \delta x^l - \delta x^l \partial_l g_{ik}. \tag{1.9.34}$$

From the equality

$$\partial_l g_{ik} = g_{ks}\Gamma^s_{il} + g_{is}\Gamma^s_{kl}, \tag{1.9.35}$$

we can write expression (1.9.34) in terms of the covariant derivatives

$$\delta_L g_{ik} = -g_{il} D_k \delta x^l - g_{kl} D_i \delta x^l. \tag{1.9.36}$$

Substituting expressions (1.9.30), (1.9.32) and (1.9.36) into the action (1.9.25), we obtain

$$\delta_c S = \frac{1}{c} \int_\Omega d^4x \left[-\delta x^l \left(\frac{\delta \mathfrak{L}}{\delta \varphi} D_l \varphi + \frac{\delta \mathfrak{L}}{\delta A_i} D_l A_i \right) - \frac{\delta \mathfrak{L}}{\delta A_i} A_l D_i \delta x^l \right.$$

$$\left. - (g_{il} D_k \delta x^l + g_{kl} D_i \delta x^l) \left(\frac{\delta \mathfrak{L}}{\delta g_{ik}} - g^{ik} g^{km} \frac{\delta \mathfrak{L}}{\delta g^{lm}} \right) + D_n J^n \right]. \tag{1.9.37}$$

We introduce the following notation

$$T^{ik} = -2 \left(\frac{\delta \mathfrak{L}}{\delta g_{ik}} - g^{il} g^{km} \frac{\delta \mathfrak{L}}{\delta g^{lm}} \right). \tag{1.9.38}$$

It will come out later, that this quantity, first introduced by Hilbert is the energy-momentum tensor for a system of fields.

Integrating by parts in (1.9.37) gives

$$\delta_c S = \frac{1}{c} \int_{\Omega} d^4x \left[-\delta x^l \left(\frac{\delta \mathfrak{L}}{\delta \varphi} D_l \varphi + \frac{\delta \mathfrak{L}}{\delta A_i} D_l A_i - D_i \left(\frac{\delta \mathfrak{L}}{\delta A_i} A_l \right) \right.\right.$$

$$\left.\left. + D_i (T^{ik} g_{lk}) + D_n \left(J^n - \frac{\delta \mathfrak{L}}{\delta A_n} A_l \delta x^l + T^{in} g_{il} \delta x^l \right) \right) \right]. \qquad (1.9.39)$$

Substituting into (1.9.26) the values of the variations $\delta_L \varphi_i$, $\delta_L A_i$, and $\delta_L g_{ik}$ according to formulas (1.9.30), (1.9.32) and (1.9.36) and combining the terms at δx^l and $D_k \delta x^l$, we have

$$J^n - \frac{\delta \mathfrak{L}}{\delta A_n} A_l \delta x^l = -\tau_l^n \delta x^l - \sigma_l^{nk} D_k \delta x^l. \qquad (1.9.40)$$

We here denoted

$$\tau_l^n = -\mathfrak{L} \delta_l^n + \frac{\partial \mathfrak{L}}{\partial(\partial_n \varphi)} D_l \varphi + \frac{\partial \mathfrak{L}}{\partial(\partial_n A_i)} D_l A_i + \frac{\partial \mathfrak{L}}{\delta A_n} A_l. \qquad (1.9.41)$$

This quantity is generally referred to as the canonic tensor of energy-momentum, and the quantity

$$\sigma_l^{nk} = 2 \frac{\partial \mathfrak{L}}{\partial(\partial_n g_{ik})} g_{il} + \frac{\partial \mathfrak{L}}{\partial(\partial_n A_k)} A_l - 2 \frac{\partial \mathfrak{L}}{\partial(\partial_n g^{lm})} g^{km} \qquad (1.9.42)$$

as the spin tensor.

Using (1.9.40) we will represent the covariant divergence in (1.9.39) as

$$D_n \left(J^n - \frac{\delta \mathfrak{L}}{\delta A_n} A_l \delta x^l + T_l^n \delta x^l \right) = \delta x^l [D_n T_l^n - D_n \tau_l^n]$$

$$+ D_n \delta x^l [T_l^n - \tau_k^n - D_k \sigma_l^{kn}] - \sigma_l^{nk} D_n D_k \delta x^l. \qquad (1.9.43)$$

Using this expression, we can write the variation (1.9.39) in the form

$$\delta_c S = \frac{1}{c} \int_{\Omega} d^4x \left[-\delta x^l \left(\frac{\delta \mathfrak{L}}{\delta \varphi} D_l \varphi + \frac{\delta \mathfrak{L}}{\delta A_i} D_l A_i - D_i \left(\frac{\delta \mathfrak{L}}{\delta A_i} A_l \right) + D_i \tau_l^i \right) \right.$$

$$\left. + D_n \delta x^l [T_l^n - \tau_l^n - D_k \sigma_l^{kn}] - \sigma_l^{nk} D_n D_k \delta x^l \right] = 0. \qquad (1.9.44)$$

Because the volume of integration is arbitrary, it follows that the inte-

grant is zero throughout, i.e.,

$$-\delta x^l\left[\frac{\delta\mathfrak{L}}{\delta\varphi}D_l\varphi+\frac{\delta\mathfrak{L}}{\delta A_i}D_lA_i-D_l\left(\frac{\delta\mathfrak{L}}{\delta A_i}A_l\right)+D_i\tau^i_l\right]$$

$$+\left[T^n_l-\tau^n_l-D_k\sigma^{kn}_l\right]D_n\delta x^l-\sigma^{nk}_l D_nD_k\delta x^l=0. \tag{1.9.45}$$

This expression vanishes for arbitrary values of δx^l, whatever the choice of the coordinate system. By virtue of the tensor law of transformation, if the expression vanishes in one system of coordinates, it does so in any other system of coordinates. It follows that

$$D_i\tau^i_l+\frac{\delta\mathfrak{L}}{\delta\varphi}D_l\varphi+\frac{\delta\mathfrak{L}}{\delta A_i}D_lA_i+D_i\left(\frac{\delta\mathfrak{L}}{\delta A_i}A_l\right)=0, \tag{1.9.46}$$

$$T^n_l-\tau^n_l-D_k\sigma^{kn}_l=0.$$

As to the last term in (1.9.45), it must vanish since the quantity σ^{nk}_l is skew-symmetric in upper indices. Since the spin tensor is skew-symmetric, we have

$$D_lT^l_n=D_l\tau^l_n. \tag{1.9.47}$$

Using (1.9.47), we can write the first identity (1.9.46) in the form

$$D_iT^i_l+\frac{\delta\mathfrak{L}}{\delta\varphi}D_l\varphi+\frac{\delta\mathfrak{L}}{\delta A_i}D_lA_i-D_i\left(\frac{\delta\mathfrak{L}}{\delta A_i}A_l\right)=0. \tag{1.9.48}$$

Identities (1.9.46) are called the strong laws of conservation; they are valid because the action is invariant under coordinate transformations.

If we take into consideration the field equations (1.9.12), we will obtain the weak conservation

$$D_kT^k_l=0,\qquad T^k_l-\tau^k_l=D_v\sigma^{pk}_l, \tag{1.9.49}$$

where the quantity τ^k_l according to the field equations and (1.9.41) will be

$$\tau^k_l=-\mathfrak{L}\delta^k_l+\frac{\partial\mathfrak{L}}{\partial(\partial_n\varphi)}D_l\varphi+\frac{\partial\mathfrak{L}}{\partial(\partial_kA_i)}D_lA_i. \tag{1.9.50}$$

The canonic energy-momentum tensor in the general case is skew-symmetric in its indices. With weak conservation of the symmetric tensor of energy-momentum the angular momentum tensor is conserved.

Defining the angular momentum tensor

$$M^{ikn}=x^kT^{in}-x^iT^{kn}, \tag{1.9.51}$$

we can easily establish, from (1.9.49) and definition (1.9.38), that

$$D_n M^{ikn} = 0. \tag{1.9.52}$$

These weak conservation laws for the tensors of energy-momentum and angular momentum do not yet suggest the conservation of energy-momentum of angular momentum for a closed system of physical fields. The existence of integral laws of conservation for a closed system is predetermined by the properties of space-time, namely with the existence of a definite group of motion of space (metric). Without the presence of a definite group of motion of space the integral laws of conservation of energy-momentum and angular momentum cannot exist. These questions are discussed in some detail in Chapter 2.

The canonic energy-momentum tensor is conveniently found in coordinates in which the metric tensor is diagonal (1, −1, −1, −1). In this case, the density of the Lagrangian has the form

$$\mathfrak{L} = \mathfrak{L}(\varphi(x), \partial_n\varphi(x), A_i(x), \partial_n A_i(x)). \tag{1.9.53}$$

The density of the Lagrangian does not depend explicitly on x, it rather depends on the coordinates through the field variables, and so we have

$$\frac{\partial \mathfrak{L}}{\partial x^l} = \frac{\delta \mathfrak{L}}{\delta \varphi}\partial_l\varphi + \frac{\partial \mathfrak{L}}{\partial(\partial_n\varphi)}\partial_l\partial_n\varphi + \frac{\partial \mathfrak{L}}{\partial A_i}\partial_l A_i$$

$$+ \frac{\partial \mathfrak{L}}{\partial(\partial_n A_i)}\partial_l\partial_n A_i. \tag{1.9.54}$$

Differentiating by parts gives

$$\frac{\partial \mathfrak{L}}{\partial x^l} = \frac{\delta \mathfrak{L}}{\delta \varphi}\partial_l\varphi + \frac{\delta \mathfrak{L}}{\delta A_i}\partial_l A_i + \partial_n\left[\frac{\partial \mathfrak{L}}{\partial(\partial_n\varphi)}\partial_l\varphi + \frac{\partial \mathfrak{L}}{\partial(\partial_n A_i)}\partial_l A_i\right]. \tag{1.9.55}$$

Applying to this identity the field equation (1.9.12) gives

$$\partial_n\left[-\mathfrak{L}\delta_l^n + \frac{\partial \mathfrak{L}}{\partial(\partial_n\varphi)}\partial_l\varphi + \frac{\partial \mathfrak{L}}{\partial(\partial_n A_i)}\partial_l A_i\right] = 0. \tag{1.9.56}$$

Under the sign of differentiation we have the canonic tensor of energy momentum in Galilean coordinates

$$\tau_l^n = -\mathfrak{L}\delta_l^n + \frac{\partial \mathfrak{L}}{\partial(\partial_n\varphi)}\partial_l\varphi + \frac{\partial \mathfrak{L}}{\partial(\partial_n A_i)}\partial_l A_i \tag{1.9.57}$$

By virtue of the stationary action principle the field equations will not change, if instead of the density of the Lagrangian $\mathfrak{L}$ we take the density

of the Lagrangian $\mathfrak{L}'$

$$\mathfrak{L}' = \mathfrak{L} + D_k f^k. \tag{1.9.58}$$

Since we only consider the Lagrangian that contains derivatives of fields of order not higher than one, the density of the vector f^k must be build up of the fields φ, A_i and the metric coefficients g_{ik}, g^{lm}. It is easily seen that

$$\frac{\delta}{\delta\varphi} D_k f^k = 0, \qquad \frac{\delta}{\delta A_i} D_k f^k = 0,$$

if we use the expression

$$D_k f^k = \partial_k f^k = \frac{\partial f^l}{\partial g_{ik}} \partial_l g_{ik} + \frac{\partial f^k}{\partial\varphi} \partial_k\varphi + \frac{\partial f^k}{\partial A_i} \partial_k A_i + \frac{\partial f^k}{\partial g^{lm}} \partial_k g^{lm}. \tag{1.9.59}$$

We will now find out how the canonic energy-momentum tensor will change under such a change in the density of the Lagrangian

$$\tau_l'^k = -\mathfrak{L}'\delta_l^k + \frac{\partial\mathfrak{L}'}{\partial(\partial_k A_i)} D_l A_i + \frac{\partial\mathfrak{L}'}{\partial(\partial_k\varphi)} D_l\varphi. \tag{1.9.60}$$

From (1.9.58) and (1.9.59), we have

$$\frac{\partial\mathfrak{L}'}{\partial(\partial_k\varphi)} = \frac{\partial\mathfrak{L}}{\partial(\partial_k\varphi)} + \frac{\partial}{\partial(\partial_k\varphi)} D_p f^p = \frac{\partial\mathfrak{L}}{\partial(\partial_k\varphi)} + \frac{\partial f^k}{\partial\varphi} \tag{1.9.61}$$

and from (1.9.61) and (1.9.57)

$$\tau_l'^k = \tau_l^k + \left[\frac{\partial f^k}{\partial A_i} D_l A_i + \frac{\partial f^k}{\partial\varphi} D_l\varphi - \delta_l^k D_m f^m\right]. \tag{1.9.62}$$

Since

$$D_l f^k = \frac{\partial f^k}{\partial A_i} D_l A_i + \frac{\partial f^k}{\partial\varphi} D_l\varphi, \tag{1.9.63}$$

we will find

$$\tau_l'^k = \tau_l^k + D_m h_l^{mk}, \tag{1.9.64}$$

where $h_l^{mk} = -h_l^{km}$ is a skew-symmetric tensor of rank three

$$h_l^{mk} = \delta_l^m f^k - \delta_l^k f^m. \tag{1.9.65}$$

Thus, if to the density of the Lagrangian we add the covariant diver-

gence, the canonic energy-momentum tensor will change by the divergence of a skew-symmetric tensor of rank three.

Let us now see how this will change the symmetric energy-momentum tensor defined according to Hilbert:

$$T^{ik} = -2\left[\frac{\delta \mathfrak{L}}{\delta g_{ik}} - g^{im} g^{kl} \frac{\delta \mathfrak{L}}{\delta g^{ml}}\right]. \tag{1.9.66}$$

Adding to the Lagrangian density of the covariant divergence will mean adding a tensor of the form

$$\Delta^{pq} = \frac{\delta}{\delta g_{pq}} D_k f^k - g^{pm} g^{ql} \frac{\delta}{\delta g^{ml}} D_k f^k. \tag{1.9.67}$$

Using (1.9.59), we will obtain, for instance,

$$\frac{\partial}{\partial g_{qp}} D_k f^k = \frac{\partial^2 f^l}{\partial g_{pq} \partial g_{ik}} \partial_l g_{ik} + \frac{\partial^2 f^l}{\partial g_{pq} \partial \varphi} \partial_l \varphi + \frac{\partial^2 f^l}{\partial g_{pq} \partial A_i} \partial_l A_i + \frac{\partial^2 f^k}{\partial g_{pq} \partial g^{mi}} \partial_k g^{ml}. \tag{1.9.68}$$

From (1.9.59), we also have

$$\frac{\partial}{\partial(\partial_m g_{pq})} D_n f^n = \frac{\partial f^m}{\partial g_{pq}}. \tag{1.9.69}$$

Using this expression gives

$$\partial_m\left(\frac{\partial}{\partial(\partial_m g_{pq})} D_n f^n\right) = \frac{\partial^2 f^m}{\partial g_{pq} \partial g_{ik}} \partial_m g_{ik} + \frac{\partial^2 f^m}{\partial g_{pq} \partial \varphi} \partial_m \varphi + \frac{\partial^2 f^m}{\partial g_{pq} \partial A_i} \partial_m A_i + \frac{\partial^2 f^m}{\partial g_{pq} \partial g^{ik}} \partial_m g^{ik}.$$

Comparing this expression with (1.9.69), we see that

$$\frac{\delta}{\delta g_{pq}} (D_n f^n) = \frac{\partial}{\partial g_{pq}} (D_n f^n) - \partial_m\left(\frac{\partial}{\partial(\partial_m g_{pq})} D_n f^n\right) = 0. \tag{1.9.70}$$

Similarly, we can show that

$$\frac{\delta}{\delta g^{ml}} (D_k f^k) = 0. \tag{1.9.71}$$

Therefore, when we add the covariant divergence to the density of the Lagrangian, this does not change the symmetric energy-momentum

tensor. The symmetric energy-momentum tensor differs from the canonic one by the divergence of the spin tensor

$$T_i^k - \tau_i^k = D_l \sigma_i^{lk} . \tag{1.9.72}$$

If the density of the Lagrangian changes by the divergence of the density of the vector, the divergence of the spin tensor will change as well, but the sum of the canonic tensor and the divergence of the spin tensor will remain unchanged. This can also be checked by direct calculation.

We will now find the expression for the symmetric energy-momentum tensor of the electromagnetic field. It is well known that the density of the Lagrangian for this field has the form

$$\mathfrak{L} = -\frac{1}{16\pi}\sqrt{-g}\, F_{lm} F^{lm}. \tag{1.9.73}$$

We write it through the variables F_{ik} and the metric coefficients

$$\mathfrak{L} = -\frac{1}{16\pi}\sqrt{-g}\, F_{ml} F_{pq}\, g^{pm} g^{ql}. \tag{1.9.74}$$

We see that the density of the Lagrangian depends on the covariant components g_{ik}, which enter into the determinant g, and on the contravariant components, which appear as two factors. Since

$$\frac{\partial\sqrt{-g}}{\partial g_{ik}} = \frac{1}{2}\sqrt{-g}\, g^{ik}, \tag{1.9.75}$$

then

$$\frac{\partial\mathfrak{L}}{\partial g_{ik}} = -\frac{1}{32\pi}\sqrt{-g}\, g^{ik} F_{lm} F^{lm}. \tag{1.9.76}$$

Similarly

$$\frac{\partial\mathfrak{L}}{\partial g^{st}} = -\frac{1}{16\pi}\sqrt{-g}\, F_{ml} F_{pq}\left[\frac{\partial g^{pm}}{\partial g^{st}} g^{lq} + g^{pm}\frac{\partial g^{lq}}{\partial g^{st}}\right].$$

Since

$$\frac{\partial g^{pm}}{\partial g^{st}} = \frac{1}{2}\left(\delta_s^p \delta_t^m + \delta_t^p \delta_s^m\right),$$

using the property that the tensor $F_{lm} = -F_{ml}$ is skew-symmetric, we will find that

$$\frac{\partial\mathfrak{L}}{\partial g^{st}} = -\frac{1}{8\pi}\sqrt{-g}\, F_{sq} F_{tl} g^{lq}. \tag{1.9.77}$$

Since the density of the Lagrangian of the electromagnetic field does not include derivatives of the metric tensor, the density of the symmetric energy-momentum tensor will be

$$T^{ik} = -2\left[\frac{\partial \mathfrak{L}}{\partial g_{ik}} - g^{is}g^{kt}\frac{\partial \mathfrak{L}}{\partial g^{st}}\right].$$

Substituting expressions (1.9.76) and (1.9.77), we obtain

$$T^{ik} = \frac{1}{4\pi}\sqrt{-g}\left[-F^{im}F^{kl}g_{ml} + \frac{1}{4}g^{ik}F_{im}F^{lm}\right]. \tag{1.9.78}$$

Dividing this by $\sqrt{-g}$, we will obtain the energy-momentum tensor of the electromagnetic field

$$t^{ik} = \frac{1}{4\pi}\left[-F^{im}F^{kl}g_{ml} + \frac{1}{4}g^{ik}F_{lm}F^{lm}\right]. \tag{1.9.79}$$

Next, we find the canonic energy-momentum tensor

$$\tau^{i}_{k} = -\delta^{i}_{k}\mathfrak{L} + \frac{\partial \mathfrak{L}}{\partial(\partial_i A_p)}\partial_k A_p. \tag{1.9.80}$$

Substituting the Lagrangian of the field, we will have

$$\tau^{i}_{k} = \frac{1}{4\pi}\left[-F^{ip}\partial_k A_p + \frac{1}{4}\delta^{i}_{k}F_{lm}F^{lm}\right]. \tag{1.9.81}$$

It is easily seen that

$$\tau^{ik} \neq \tau^{ki}. \tag{1.9.82}$$

The spin tensor of the electromagnetic field is

$$\sigma^{pi}_{k} = \frac{\partial \mathfrak{L}}{\partial(\partial_p A_i)}A_k = -\frac{1}{4\pi}F^{pi}A_k. \tag{1.9.83}$$

Adding the spin divergence to the canonic tensor, we will obtain the symmetric energy-momentum tensor of the electromagnetic field (according to Hilbert)

$$t^{i}_{k} = \tau^{i}_{k} + \partial_p\sigma^{pi}_{k} = \frac{1}{4\pi}\left[-F^{ip}F_{kp} + \frac{1}{4}\delta^{i}_{k}F_{lm}F^{lm}\right] \tag{1.9.84}$$

or, multiplying both parts of this by g^{kn}, we will have

$$t^{in} = \frac{1}{4\pi}\left[-F^{ip}F^{nl}g_{pl} + \frac{1}{4}g^{in}F_{lm}F^{lm}\right], \tag{1.9.85}$$

which agrees with (1.9.78).

Before we leave the section, we will construct the tensor of energy-momentum of matter. As is known from Section 1.8, the density of the mass conserved is

$$\mu = \frac{dm}{dV} = \sqrt{-g}\,\mu_0 u^0 \frac{1}{c}. \tag{1.9.86}$$

Since the quantity dm/dV is independent of the metric tensor, and the four-dimensional velocity u^i is

$$u^i = \frac{cv^i}{\sqrt{g_{lk}v^l v^k}}, \qquad v^i = \frac{dx^i}{dt}, \tag{1.9.87}$$

the variation of (1.9.86) in the metric tensor will be zero, i.e.,

$$\delta_g \mu = u^0 \delta_g(\sqrt{-g}\,\mu_0) - \sqrt{-g}\,\mu_0 \frac{c^2 v^i v^k \delta g_{ik}}{2(g_{mn}v^m v^n)^{3/2}} = 0. \tag{1.9.88}$$

It follows that

$$\delta_g(\sqrt{-g}\,\mu_0) = \sqrt{-g}\,\mu_0 \frac{u^i u^k}{2c^2}\delta g_{ik}. \tag{1.9.89}$$

Since the density of the Lagrangian of the action of matter is

$$\mathfrak{L} = -\sqrt{-g}\,\mu_0 c^2,$$

the density of the energy-momentum tensor for matter will be

$$t_c^{ik} = -2\frac{\partial \mathfrak{L}}{\partial g_{ik}} = \sqrt{-g}\,\mu_0 u^i u^k. \tag{1.9.90}$$

Considering this and also the formula (1.9.85), we will write the total tensor of the energy-momentum of matter and electrical field as

$$t^{ik} = \mu_0 u^i u^k + \frac{1}{4\pi}\left[-F^{ip}F^{kl}g_{pl} + \frac{1}{4}g^{ik}F_{lm}F^{lm}\right]. \tag{1.9.91}$$

1.10. Belinfante Energy-Momentum Tensor

In Section 1.9 we have constructed the Hilbert energy-momentum tensor using infinitesimal displacements δx^l in an arbitrary curvilinear coordinate system. This way of constructing the energy-momentum tensor is general; however, it is rather complicated. The problem of constructing the energy-momentum tensor can be simplified, if we confine ourselves to the displacements δx_i that obey the Killing equation

$$\delta_L g_{ik} = -D_k \delta x_i - D_i \delta x_k = 0.$$

Since space-time is pseudo-Euclidean, this can always be done. Moreover, we can always construct all 10 independent Killing vectors and use them to find the integral conservation laws. The conservation laws for energy-momentum and angular momentum for a closed system are a direct consequence of the existence of 10 Killing vectors. For these displacements, from (1.9.25) and (1.9.26), we find

$$\delta_c S = \frac{1}{c} \int d^4x \left(\frac{\delta \mathfrak{L}}{\delta \varphi} \delta_L \varphi + \frac{\delta \mathfrak{L}}{\delta A_i} \delta_L A_i + D_n J^n\right), \tag{1.10.1}$$

$$J^n = \mathfrak{L} \delta x^n + \frac{\partial \mathfrak{L}}{\partial(\partial_n \varphi)} \delta_L \varphi + \frac{\partial \mathfrak{L}}{\partial(\partial_n A_i)} \delta_L A_i. \tag{1.10.2}$$

If the fields φ and A_i obey the field equations (1.9.12), then

$$\delta_c S = \frac{1}{c} \int_\Omega d^4x (D_n J^n) = 0. \tag{1.10.3}$$

Because the volume of integration is arbitrary, we thus find

$$D_n J^n = 0. \tag{1.10.4}$$

Substituting into (1.10.2) the expression (1.9.30) and (1.9.32) for the Lie variations $\delta_L A_i$ and $\delta_L \varphi$, we find

$$J^n = -\delta x^l \tau_l^n - \tilde{\sigma}_l^{nk} D_k \delta x^l, \tag{1.10.5}$$

where the canonic energy-momentum tensor has the form

$$\tau_l^n = -\mathfrak{L} \delta_l^n + \frac{\partial \mathfrak{L}}{\partial(\partial_n \varphi)} D_l \varphi + \frac{\partial \mathfrak{L}}{\partial(\partial_n A_i)} D_l A_i, \tag{1.10.6}$$

and the spin tensor is

$$\tilde{\sigma}_l^{nk} = \frac{\partial \mathfrak{L}}{\partial(\partial_n A_k)} A_l. \tag{1.10.7}$$

Substituting the expression for the current (1.10.5) into (1.10.4) gives

$$D_n J^n = -\delta x_l D_n \tau^{nl} - [\tau^{nl} + D_k \tilde{\sigma}^{l;kn}] D_n \delta x_l$$

$$- \tilde{\sigma}^{l;nk} D_n D_k \delta x_l = 0. \tag{1.10.8}$$

By the Killing equations, we have

$$D_n D_k \delta x_l = 0. \tag{1.10.9}$$

In a Galilean coordinate system, (1.10.8) has the simplest form

$$-\delta x_l \partial_n \tau^{nl} - [\tau^{nl} + \partial_k \tilde{\sigma}^{l;kn}] \partial_n \delta x_l = 0. \tag{1.10.10}$$

In a Galilean coordinate system, the Killing equations

$$\partial_k \delta x_i + \partial_i \delta x_k = 0 \tag{1.10.11}$$

have the general solution in the form of a linear function of x^l

$$\delta x_i = \epsilon_i + w_{il} x^l, \tag{1.10.12}$$

where ϵ_i is a constant vector, and w_{il} is the skew-symmetric matrix of order four

$$w_{il} + w_{li} = 0. \tag{1.10.13}$$

Substituting into (1.10.10) the displacement of the form

$$\delta x_l = \epsilon_l, \tag{1.10.14}$$

we have

$$\partial_n \tau^{nl} = 0. \tag{1.10.15}$$

Similarly, for displacements of the type

$$\delta x_l = w_{lm} x^m \tag{1.10.16}$$

we have

$$(\tau^{nl} + \partial_k \tilde{\sigma}^{l;kn}) w_{ln} = 0. \tag{1.10.17}$$

The tensor w_{ln} being skew-symmetric and arbitrary, we find that the quantity

$$\tau^{nl} + \partial_k \tilde{\sigma}^{l;kn} \tag{1.10.18}$$

is symmetric in indices n and l. Hence

$$\tau^{nl} - \tau^{ln} = -\partial_k (\tilde{\sigma}^{l;kn} - \tilde{\sigma}^{n;kl}). \tag{1.10.19}$$

Introduce the notation

$$H^{nlk} = \tilde{\sigma}^{l;kn} - \tilde{\sigma}^{n;kl} = -H^{lnk}. \tag{1.10.20}$$

We now find the quantity that is antisymmetric in indices k and n

$$\sigma^{l,kn} = \frac{1}{2}(H^{nlk} + H^{nkl} - H^{kln}). \tag{1.10.21}$$

According to Belinfante, we will then find

$$T^{nl} = \tau^{nl} + \frac{1}{2}\partial_k(H^{nlk} + H^{nkl} - H^{kln}). \tag{1.10.22}$$

Substituting the expression (1.10.7) for the spin tensor into the definition (1.10.20), we obtain

$$H^{nlk} = \frac{\partial \mathfrak{L}}{\partial(\partial_k A_n)} A^l - \frac{\partial \mathfrak{L}}{\partial(\partial_k A_l)} A^n. \tag{1.10.23}$$

It can easily be seen that by (1.10.19) and (1.10.20) the energy-momentum tensor obeys the conservation law

$$\partial_n T^{nl} = 0, \qquad \partial_l T^{nl} = 0 \tag{1.10.24}$$

and is symmetric.

We now show that the Belinfante tensor is nothing else but the Hilbert energy-momentum. The Hilbert tensor can be found either by computing the Euler variation in metric coefficients (to this end, it is necessary to write the density of the Lagrangian in the curvilinear coordinate system), or, using the expression for the canonic energy-momentum tensor and the spin tensor, by formula (1.9.49). The second technique can be realized while still remaining in a Galilean coordinate system. The Belinfante method, as we will see, is the construction of the Hilbert tensor in this way. Indeed, we have earlier obtained in (1.9.47) and (1.9.49) the weak conservation laws, which, if we raise the indices on the tensors and change to a Galilean coordinate system, can be written as

$$\partial_k T^{kl} = 0, \quad T^{kl} - \tau^{kl} = \partial_p \sigma^{l;pk}, \quad \partial_k \tau^{kl} = 0. \tag{1.10.25}$$

The Hilbert energy-momentum tensor T^{kl} is symmetrical by the definition (1.9.38). By the second equality the quantity

$$\tau^{kl} + \partial_p \sigma^{l;pk} = T^{kl} \tag{1.10.26}$$

is symmetric as well, and hence

$$\tau^{kl} - \tau^{lk} = -\partial_p(\sigma^{l;pk} - \sigma^{k;pl}). \tag{1.10.27}$$

Denoting

$$H^{klp} = \sigma^{l;pk} - \sigma^{k;pl}, \tag{1.10.28}$$

we find

$$\sigma^{l;pk} = \frac{1}{2} (H^{klp} + H^{kpl} - H^{plk}). \tag{1.10.29}$$

Substituting this expression into (1.10.26), we will represent the Hilbert tensor in the Belinfante form

$$T^{kl} = \tau^{kl} + \frac{1}{2} \partial_p [H^{klp} + H^{kpl} - H^{plk}]. \tag{1.10.30}$$

In a Galilean system of coordinates the expression (1.9.41) for the canonic energy-momentum tensor has the form

$$\tau_l^k = -\mathfrak{L}\delta_l^k + \frac{\partial \mathfrak{L}}{\partial(\partial_k \varphi)} \partial_l \varphi + \frac{\partial \mathfrak{L}}{\partial(\partial_k A_i)} \partial_l A_i. \tag{1.10.31}$$

Comparing (1.10.22) with (1.10.30) and taking account of (1.10.31) and (1.10.23), we see that the Belinfante tensor is nothing else but the Hilbert tensor in a Galilean coordinate system.

Let us come back to the foundations of the theory of relativity. Unfortunately, there is much confusion in the literature concerning these issues. It is our aim here to clarify the picture and at the same time to expand the applicability of the special theory of relativity. We have already broached some of the questions earlier in the book.

1.11. Coordinate Velocity of Light

We will now determine the coordinate velocity of light. It is well known that the motion of a light signal is described by the isotropic interval

$$ds^2 = 0.$$

Introducing the notation

$$v = \frac{dx^\alpha}{dt}$$

for the coordinate velocity of light and considering that the tensor indices of this velocity are raised and lowered using the tensor $\kappa_{\alpha\beta}$ (1.4.14)

$$v_\alpha = \kappa_{\alpha\beta} v^\beta, \quad v^2 = \kappa_{\alpha\beta} v^\alpha v^\beta,$$

we will represent the interval for this case as

$$c^2 \left[\sqrt{g_{00}} + \frac{g_{0\alpha} v^\alpha}{c\sqrt{g_{00}}} \right]^2 - v^2 = 0. \tag{1.11.1}$$

Writing the velocity vector v^α in the form

$$v^\alpha = ve^\alpha, \tag{1.11.2}$$

where e^α is the unit vector in the direction of the velocity v^α,

$$\kappa_{\alpha\beta}e^\alpha e^\beta = 1, \tag{1.11.3}$$

we will have by (1.5.1)

$$v = c\left[\sqrt{g_{00}} + \frac{g_{0\alpha}e^\alpha}{\sqrt{g_{00}}}\frac{v}{c}\right].$$

Solving this equation for v, we obtain

$$v = \frac{c\sqrt{g_{00}}}{1 - \dfrac{g_{0\alpha}e^\alpha}{\sqrt{g_{00}}}}. \tag{1.11.4}$$

The magnitude of the coordinate velocity is thus dependent on the metric coefficients and also on the direction; the expression in the denominator (1.11.4) is always positive and it does not vanish.

To see this, we multiply $\kappa_{\alpha\beta}$ by $e^\alpha e^\beta$. By the definition (1.11.3), this is unity. On the other hand, we will have

$$1 = \kappa_{\alpha\beta}e^\alpha e^\beta = -g_{\alpha\beta}e^\alpha e^\beta + \left[\frac{g_{0\alpha}e^\alpha}{\sqrt{g_{00}}}\right]^2. \tag{1.11.5}$$

For permissible coordinate systems, the quadratic form $g_{\alpha\beta}dx^\alpha dx^\beta$, and hence the quadratic form $g_{\alpha\beta}e^\alpha e^\beta$, is negative definite

$$g_{\alpha\beta}e^\alpha e^\beta < 0. \tag{1.11.6}$$

From (1.11.5) and (1.11.6) we will then get

$$\left[\frac{g_{0\alpha}e^\alpha}{\sqrt{g_{00}}}\right]^2 = 1 + g_{\alpha\beta}e^\alpha e^\beta < 1.$$

In consequence, the denominator in the expression (1.11.4) never vanishes.

When the metric coefficients g_{ik} are constant, the velocity v will also be constant, though different for various directions of motion in space. It was using this definition of the light velocity that Einstein, Pauli, Reichenbach, Mandelshtam, and others, accomplished clock synchronization at different points of space in the inertial frame of reference. We see that the synchronization, defined in this manner,

depends on the choice of a coordinate system of space-time in an inertial frame of reference, and therefore is an event of no physical importance whatsoever. In Galilean coordinates this velocity is c and does not depend on the direction of the motion in space. Stressing this circumstance, Pauli wrote about the postulate of the constancy of the velocity of light [19]: "Any *universal* constancy of the velocity of light in a vacuum is out of the question, if only because the velocity of light is only constant in Galilean frames of reference*." But this conclusion is based on the definition, as we now see, of the coordinate velocity of light, rather than the physical velocity of light. Now then, we cannot make any inferences from nonphysical concepts. How can one possibly define the concept of physical velocity, specifically, of the physical velocity of light?

To do so, we will need to know, first, the distance between points A and B in space and, second, the length of time needed for a signal to come to B from A. But to determine both requires a knowledge of space and time or, more precisely, a knowledge of the geometry of space-time, which only became possible after Minkowski had discovered his pseudo-Euclidean geometry of space-time. Up to now, however, Minkowski's discovery has been interpreted formally or misunderstood, which in turn obscured relativity theory. Testimony of this is to be found in a host of textbooks, monographs and treatments of various authors.

We have earlier (see (1.4.12)) derived an expression of physical time in terms of the coordinate variables of space-time

$$d\tau = dt\sqrt{g_{00}} + \frac{g_{0\alpha}dx^{\alpha}}{c\sqrt{g_{00}}}.$$

Similarly, we have expressed (See (1.4.13)) the infinitesimal distance between two points in space through space variables

$$dl^2 = \left(-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right)dx^{\alpha}dx^{\beta}.$$

Hence the physical velocity is

$$V = \frac{dl}{d\tau}.$$

For a light signal the interval is zero, and so the physical velocity of light in an inertial frame of reference in any permissible systems of

*Here Pauli, following Einstein, has in mind Galilean coordinates.

coordinates of space-time is always c and is independent of the direction of motion, i.e.,

$$V = \frac{dl}{d\tau} = c.$$

It follows that the postulate of the constancy of the velocity of light, as we have established, is valid in an inertial frame of reference always, irrespective of the choice of permissible coordinates of space-time. The fallacy was thus due to the fact that the treatment dealt with the coordinate velocity of light, not the physical velocity of light. In the special case of Galilean coordinates the coordinate velocity of light coincides with the physical one. In any other coordinates in an inertial reference frame the physical velocity of light is always dissimilar with the coordinate one. The latter, generally speaking, can be as large as you like.

The definition of the physical velocity of light which follows from the pseudo-Euclidean structure of the geometry of space-time enables us uniquely to synchronize clocks at different points of space in an inertial reference frame in any permissible coordinates of space-time. It is the introduction of the physical velocity of light instead of the coordinate one, which was nearly always involved in calculations, that completely removes all questions of ambiguity that have allegedly emerged earlier in describing physical events. And so the postulate of the constancy of the velocity of light can be formulated in an inertial frame of reference in any permissible coordinates of space-time as a special consequence of the pseudo-Euclidean structure of space-time on the basis the concept of the physical velocity we have just introduced. But even formulated this way, this postulate has a limited meaning since in an arbitrary (accelerated) reference frame it does not hold true, although physical phenomena can be described in any reference frame, since relativity theory is a theory of space-time. It is thus obvious that the postulates of the constancy of the velocity of light in principle do not allow us to venture beyond the domain of inertial reference frames in the special theory of relativity.

On the other hand, the idea of the pseudo-Euclidean geometry of space-time is more general and fundamental. It affords a unified approach to the formulation of physical laws in inertial and noninertial (accelerated) systems alike, thereby extending the scope of the special theory of relativity. This is however not only of fundamental but also of applied value. Let us trace the history of the question.

In 1907 Einstein [8] analyzed gravitation: "... in what follows we will also assume the total physical equivalence of gravitational field

and the corresponding acceleration of a frame of reference." He goes on the expand on the idea, which he formulated as the equivalence principle, and writes in 1913 [8]: "Conventional Relativity only permits linear orthogonal transformation." In his next paper of the same year [8] he writes: "In the original relativity theory the independence of physical equations of the special choice of a frame of reference is based on postulating the fundamental invariant $ds^2 = \sum_i dx_i^2$, and we have to construct a theory (he means the general theory of relativity – *A.L.*) where the role of the fundamental invariant is played by the linear element of the most general type $ds^2 = \sum_{i,k} g_{ik} dx_i dx_k$." Einstein proclaims the quantities g_{ik} to be characteristics of the gravitational field. These views of Einstein's on the special theory of relativity narrow down its scope dramatically. This alone goes to prove that Einstein failed to grasp the profound physical content of Minkowski's discovery.

As we know, Poincaré and Minkowski discovered pseudo-Euclidean geometry and established the unit of space and time. This suggests that the interval between events in an arbitrary permissible system of coordinates of space-time can in the general form be written as

$$ds^2 = \sum_{i,k} g_{ik} dx^i dx^k .$$

In the special case of Galilean coordinates we have

$$s_{12}^2 = c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 ,$$

or, in the differential form,

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 .$$

And so in the special theory of relativity we no longer deal with postulation of the interval in the form

$$ds^2 = \sum_i dx_i^2 ,$$

as Einstein believed, but rather with the pseudo-Euclidean geometry of space-time given by the interval

$$ds^2 = g_{ik} dx^i dx^k$$

with the metric tensor g_{ik} for which the Riemannian curvature tensor R_{iklm} is zero. Why then has Einstein failed to see this? This seems to be due to the fact that he perceived the special theory of relativity only through the postulate of the constancy of the velocity of light in

Galilean coordinates, and he identified accelerated reference frames with gravitation on the basis of the equivalence principle. Even in the current literature we often encounter misunderstandings of Minkowski's discovery, and hence of the essence of the special theory of relativity. Without creative assessment of even great inheritance of the past science simply cannot exist. This truth is perfectly well known, though not always followed, since this requires much time and effort. After all, it is simpler just to retell than to rework. It is especially important to study fundamental works, because in them one can trace the inception and evolution of great ideas.

As I have already mentioned, one should generally distinguish two types of quantities: coordinate and physical ones. Coordinate quantities are sensitive to the choice of the coordinate-time grid used to describe an event. They are defined by a definite measuring technique. Physical quantities are quantities that are objective characteristics of space, time and matter. Therefore, to construct physical quantities, along with coordinate quantities it is necessary to employ the metric tensor of space-time, which permits to get rid of arbitrariness in the choice of coordinates. So, for instance, the physical quantities $d\tau$ and dl^2, given by (1.4.12) and (1.4.13), are independent of the choice of a coordinate system in a given inertial reference frame, since they are invariant with respect to a group of transformations (1.9.1).

It is to be noted here that when describing a phenomenon we for the most part chose the coordinates (x, y, z, t) arbitrarily, based on the choice of a coordinate-time grid. In other words, the choice is based on the arithmetization of points in space-time, just as on a conventional plane we can introduce various systems of coordinates: oblique-angled, curvilinear, or even Gaussian.

It should be stressed, however, that the procedure of constructing a coordinate grid, i.e., placing into correspondence to each point in space-time of a set of four numbers (x, y, z, t) is an operational one. It presupposes an arbitrary, although definite, choice of the physical way of arithmetization of space-time.

It is important to note that, although the coordinates are chosen in an arbitrary manner, they must be permissible, i.e., they must allow the corresponding coordinate system to be realized using real physical processes. To do so, it is necessary that the component g_{00} of the metric tensor be positive, and the quadratic form constructed using the space components $g_{\alpha\beta}$ of the metric tensor be negatively definite, i.e.,

$$\begin{aligned} &g_{00} > 0, \\ &g_{\alpha\beta}dx^{\alpha}dx^{\beta} < 0. \end{aligned} \tag{1.11.7}$$

For the latter condition to hold, it is necessary and sufficient, by Sylvester's criterion, that

$$g_{11} < 0, \quad \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} < 0. \tag{1.11.8}$$

We also note that, by conditions (1.11.7), the quadratic form

$$\kappa_{\alpha\beta} dx^\alpha dx^\beta = -g_{\alpha\beta} dx^\alpha dx^\beta + \frac{(g_{0\alpha} dx^\alpha)^2}{g_{00}} \tag{1.11.9}$$

will at all times be positive definite, and so the determinants

$$\Delta_1 = \kappa_{11}, \quad \Delta_2 = \begin{vmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{12} & \kappa_{22} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & \kappa_{22} & \kappa_{23} \\ \kappa_{13} & \kappa_{23} & \kappa_{33} \end{vmatrix}$$

will also be always positive, i.e., $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$. This enables us to carry out the following transformations

$$dx'^1 = \frac{\kappa^1_\alpha dx^\alpha}{\sqrt{\Delta_1}},$$

$$dx'^2 = \sqrt{\frac{\Delta_2}{\Delta_1}}\, dx^2 + \frac{(\kappa_{23}\kappa_{11} - \kappa_{12}\kappa_{13})}{\sqrt{\Delta_2 \Delta_1}}\, dx^3,$$

$$dx'^3 = \sqrt{\frac{\Delta_3}{\Delta_2}}\, dx^3,$$

where dx'^α in a Riemannian space are not total differentials. This transformation reduced the quadratic form (1.11.9) to diagonal form

$$dl^2 = \kappa_{\alpha\beta} dx^\alpha dx^\beta = (dx'^1)^2 + (dx'^2)^2 + (dx'^3)^2.$$

To summarize, when we assumed that unified space-time features pseudo-Euclidean geometry and considered coordinate systems accordingly, we could cast a new glance at the principle of relativity and, most important, substantially expand the applicability of the special theory of relativity.

1.12. Generalized Inertial Reference Frames

Let us consider now an arbitrary linear transformation from Galilean coordinates (X, Y, Z, T) to some coordinates (x, y, z, t)

$$\begin{aligned} T &= a_0 t + a_1 x + a_2 y + a_3 z, \\ X &= a_4 t + a_5 x + a_6 y + a_7 z, \\ Y &= a_8 t + a_9 x + a_{10} y + a_{11} z, \\ Z &= a_{12} t + a_{13} x + a_{14} y + a_{15} z. \end{aligned} \tag{1.12.1}$$

Without loss of generality, we require that the space axes of the new system (x, y, z) be orthogonal to one another. Then, up to a turn of the space coordinate system, transformation (1.12.1) will be equivalent to the transformation

$$\begin{aligned} T &= qx + pt, \quad & X &= ax + bt, \\ Y &= y, & Z &= z, \end{aligned} \tag{1.12.2}$$

where a, b, p, q are some constants.

Taking differentials of the right- and left-hand sides of the relations and substituting into (1.2.13), we get

$$ds^2 = c^2 g_{00} dt^2 + 2c g_{01} dt dx + g_{11} dx^2 - dy^2 - dz^2, \tag{1.12.3}$$

where

$$\begin{aligned} g_{00} &= p^2 - \frac{b^2}{c^2}, \quad g_{01} = c\left(pq - \frac{ab}{c^2}\right), \\ g_{11} &= c^2 q^2 - a^2, \quad g_{22} = g_{33} = -1. \end{aligned} \tag{1.12.4}$$

For the transformation (1.12.2) to be permissible, we must ensure that the following conditions are met:

$$p^2 - \frac{b^2}{c^2} > 0, \quad c^2 q^2 - a^2 < 0. \tag{1.12.5}$$

Metric (1.12.3) desctibes an arbitrary inertial frame with coordinates that are slightly different from Galilean ones.

We now define the coordinate velocity of a light wave. For the wave the interval is zero.

We now introduce the notation

$$v^x = \frac{dx}{dt}, \quad v^y = \frac{dy}{dt}, \quad v^z = \frac{dz}{dt}$$

for the components of the coordinate velocity of the light signal. Expres-

sion (1.12.3) then becomes

$$g_{11}(v^x)^2 + 2cg_{01}v^x + c^2 g_{00} - (v^y)^2 - (v^z)^2 = 0.$$

Solving this quadratic equation for v^x gives

$$v^x = c\left[\frac{-g_{01} \pm \sqrt{g_{01}^2 - g_{11}\left[g_{00} - \dfrac{(v^y)^2 + (v^z)^2}{c^2}\right]}}{g_{11}}\right].$$

We have thus two roots: one positive and one negative. If the light signal travels along the x-axis, the expression is simplified

$$\begin{aligned} c_1 &= c\,\frac{-g_{01} - \sqrt{g_{01}^2 - g_{00}g_{11}}}{g_{11}} > 0; \\ c_2 &= c\,\frac{-g_{01} + \sqrt{g_{01}^2 - g_{00}g_{11}}}{g_{11}} < 0, \end{aligned} \tag{1.12.6}$$

where c_1 is the coordinate velocity of light along the positive x-axis, c_2 is the coordinate velocity of light in the opposite direction.

The magnitude of the coordinate velocity of light, as we have seen, is defined by (1.11.4), and the unit vector e^α obeys (1.11.3). In the case at hand the components of the metric tensor that enter into these expressions are

$$g_{02} = 0, \quad g_{03} = 0, \quad \kappa_{11} = -g_{11} + \frac{g_{01}^2}{g_{00}} > 0,$$

$$g_{01} \neq 0, \quad g_{00} > 0, \quad \kappa_{22} = 1, \quad \kappa_{33} = 1, \quad \kappa_{\alpha\beta} = 0, \quad \alpha \neq \beta.$$

Substituting these into (1.11.3) gives

$$\kappa_{11}(e^1)^2 + (e^2)^2 + (e^3)^2 = 1.$$

Therefore, the unit vector e^α for the velocity of light will have the components

$$e^1 = \frac{\sin\theta\cos\varphi}{\sqrt{\kappa_{11}}}, \quad e^2 = \sin\theta\sin\varphi, \quad e^3 = \cos\theta. \tag{1.12.7}$$

From (1.11.4) we can now find the magnitude of the coordinate velocity of light to be

$$v = \frac{c\sqrt{g_{00}}}{1 - \dfrac{g_{01}}{\sqrt{g_{00}\kappa_{11}}}\sin\theta\cos\varphi}. \tag{1.12.8}$$

Since the quantity $g_{01}/\sqrt{g_{00}\kappa_{11}}$ is always less than unity, in the space of velocities the expression (1.12.8) describes an ellipsoid of revolution (see Fig. 2). The cross-section of the ellipsoid by the xy-plane ($\theta = \pi/2$) has the shape of the ellipse

$$v = \frac{c\sqrt{g_{00}}}{1 - \dfrac{g_{01}}{\sqrt{g_{00}\kappa_{11}}}\cos\varphi}$$

with the eccentricity $e = g_{01}/\sqrt{g_{00}\kappa_{11}} < 1$ and the parameter $p = c\sqrt{g_{00}}$; and the cross-section by the yz-plane ($\varphi = \pi/2$) is the circle of radius $v = c\sqrt{g_{00}}$.

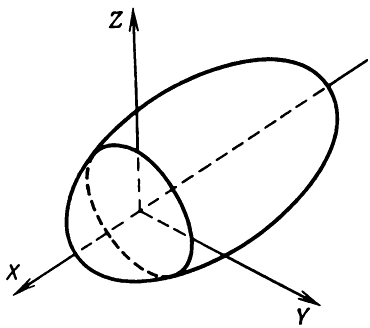

Fig. 2. The ellipsoid of the coordinate velocity of light

Therefore the magnitude of the coordinate velocity of light in any direction is determined by the metric of space-time.

In the general case of an inertial coordinate system we have

$$c_1 \neq -c_2,$$

i.e., the velocity of light along the positive x-axis is unequal to the velocity of light in the opposite direction.

It follows from (1.12.6) that

$$c_1 + c_2 = -2c\,\frac{g_{01}}{g_{11}}, \quad c_1 c_2 = c^2\,\frac{g_{00}}{g_{11}}.$$

Hence

$$g_{11} = \frac{c^2}{c_1 c_2}\,g_{00}, \quad 2g_{01} = -cg_{00}\left(\frac{1}{c_1} + \frac{1}{c_2}\right). \tag{1.12.9}$$

Therefore, by expressions (1.12.7)–(1.12.9), the components of the

coordinate velocity of light $v^\alpha = ve^\alpha$ can be written as

$$v^1 = \frac{2c_1 c_2 \sin\theta \cos\varphi}{c_2 - c_1 + (c_2 + c_1)\sin\theta\cos\varphi},$$

$$v^2 = \frac{c\sqrt{g_{00}}\sin\theta\sin\varphi}{1 + \dfrac{c_2 + c_1}{c_2 - c_1}\sin\theta\cos\varphi},$$

$$v^3 = \frac{c\sqrt{g_{00}}\cos\theta}{1 + \dfrac{c_2 + c_1}{c_2 - c_1}\sin\theta\cos\varphi}. \tag{1.12.10}$$

We can see from (1.12.10) that the values of the coordinate velocity of light in the forward ($\theta = \theta_0$, $\varphi = \varphi_0$) and back ($\theta = \pi - \theta_0$, $\varphi = \varphi_0 + \pi$) directions are in the general case unequal. In particular, at $\theta_0 = \pi/2$, $\varphi_0 = 0$ in the forward direction we have $v^1 = c_1$, and in the opposite direction $v^1 = c_2$.

Making use of the relations (1.12.9) between various components of the metric tensor we can write the expression (1.12.3) for the interval as

$$ds^2 = c^2 g_{00}\left[dt^2 - \left(\frac{1}{c_1} + \frac{1}{c_2}\right)dx dt + \frac{dx^2}{c_1 c_2}\right] - dy^2 - dz^2. \tag{1.12.11}$$

Interval (1.12.11) describes a sufficiently arbitrary inertial system, in which the coordinate velocities of light in opposite directions are in the general case unequal. For the case of the equality $c_1 = -c_2 = c$, the interval (1.12.11) assumes the Galilean form (1.2.13).

To sum up: in an arbitrary interval reference frame Einstein's postulate of the constancy of the velocity of light (as formulated by him) does not hold, but nevertheless this circumstance has no influence on the possibility of describing physical reality in these frames. To be sure, knowing the pseudo-Euclidean structure of space-time, as we have seen, we can give a general formulation of the postulate of the constancy of the velocity of light in an inertial frame of reference and for any permissible coordinates of space-time.

1.13. Transformations Between Different Generalized Inertial Frames

Consider some inertial frame of reference in which the interval has the form (1.12.11). We will now find an eigengroup of coordinate transformations such that they leave the metric form-invariant.

As we know, in the case of the Galilean metric (1.12.13) this eigengroup, i.e., a group of coordinate transformations with a Jacobian equal to +1, is a ten-parameter one and includes three eigensubgroups, namely a four-parameter subgroup of translations of the time and space coordinates, a three-parameter subgroup of three-dimensional turns of the coordinate system and a three-parameter subgroup of Lorentzian rotations.

We now show that in the case of the metric (12.11) there will also be a ten-parameter group of inertial frames of reference, such that in them the coordinate velocity of light is unequal to c and is direction-dependent. What is more, transformations corresponding to a transition from one inertial frame to another in this group leave the metric form-invariant. Accordingly, all the equations in physics will in any of these frames have the same functional dependence on coordinates; therefore, no physical experiment can tell us in which of a given infinite collection of frames we reside at the moment. Description of various physical processes in these inertial frames yields results that coincide with experiment.

To begin with, we determine the three-parameter subgroup of transformations that corresponds to the transition between different inertial frames of reference and that leave the metric form-invariant.

Since in expressions (1.12.11) for the interval the x-axis is distinguished, one should expect that the desired transformation would have the simplest form when the relative velocity of two inertial frames of reference is along the x-axis. Therefore, we first consider this special case, for it enables us, using the least cumbersome calculations, to show in the most graphic manner all the possibilities available here. A generalization to the case where the relative velocity of two inertial frames has an arbitrary direction is achieved trivially and provides nothing radically new.

It is quite obvious that in this case the required transformation is linear:

$$x_{\text{old}} = ax_{\text{new}} + bt_{\text{new}}, \qquad t_{\text{old}} = qx_{\text{new}} + pt_{\text{new}}$$
$$y_{\text{old}} = y_{\text{new}}, \qquad z_{\text{old}} = z_{\text{new}}. \tag{1.13.1}$$

The requirement that the metric (1.12.11) be form-invariant under transformation (1.13.1) implies that under this transformation the

metric (1.12.11) must become

$$ds^2 = c^2 g_{00}\left[dt_{\text{new}}^2 - \left(\frac{1}{c_1} + \frac{1}{c_2}\right)dt_{\text{new}}dx_{\text{new}} + \frac{dx_{\text{new}}^2}{c_1c_2}\right]$$

$$- dy_{\text{new}}^2 - dz_{\text{new}}^2. \tag{1.13.2}$$

On taking differential of both sides of (1.13.1), we substitute them into the expression (1.12.11). A comparison of the resultant expression with metric (1.13.2) shows that the requirement of form-invariance is equivalent to the conditions that

$$\begin{aligned} &\Phi_1\Phi_1 = 1, \quad Q_1Q_2 = \frac{1}{c_1c_2}, \\ &Q_1\Phi_2 + Q_2\Phi_1 = -\left(\frac{1}{c_1} + \frac{1}{c_2}\right), \end{aligned} \tag{1.13.3}$$

where we have introduced the notation

$$\begin{aligned} &Q_1 = q - \frac{a}{c_1}, \quad Q_2 = q - \frac{a}{c_2}, \quad \Phi_1 = p - \frac{b}{c_1}, \\ &\Phi_2 = p - \frac{b}{c_2}. \end{aligned} \tag{1.13.4}$$

The solution of the system (1.13.3) corresponding to the transformations with a Jacobian equal to +1 has the form

$$\Phi_2 = \frac{1}{\Phi_1}, \quad Q_1 = -\frac{\Phi_1}{c_1}, \quad Q_2 = -\frac{1}{c_2\Phi_1}. \tag{1.13.5}$$

Finding from (1.13.4) and (1.13.5) the parameters a, b, p, q and substituting them into (1.13.1), we will have

$$x_{\text{old}} = \frac{(c_1 - c_2\Phi_1^2)x_{\text{new}} + c_1c_2(\Phi_1^2 - 1)t_{\text{new}}}{\Phi_1(c_1 - c_2)},$$

$$t_{\text{old}} = \frac{(1 - \Phi_1^2)x_{\text{new}} + (c_1\Phi_1^2 - c_2)t_{\text{new}}}{\Phi_1(c_1 - c_2)}. \tag{1.13.6}$$

Transformations (1.13.6) must describe a transition from one inertial frame to another, and so we have to require that some point in the "old" frame (e.g., the origin of coordinates) and the "new" frame would move with the coordinate velocity v against the x-axis according to the law $x = -vt$.

Substituting into the first of (1.13.6) $x_{\text{old}} = 0$ and $x_{\text{new}} = -vt_{\text{new}}$, we will get

$$\Phi_1^2 = \frac{c_1(v + c_2)}{c_2(v + c_1)} .$$

Transformations (1.13.6) will then become

$$x_{\text{old}} = \frac{x_{\text{new}} + vt_{\text{new}}}{\sqrt{(1 - v/c_1)(1 + v/c_2)}},$$

$$t_{\text{old}} = \frac{\left(1 + \frac{v}{c_1} + \frac{v}{c_2}\right) - \frac{v}{c_1 c_2} x_{\text{new}}}{\sqrt{(1 + v/c_1)(1 + v/c_2)}},$$

$$y_{\text{old}} = y_{\text{new}}, \quad z = z_{\text{new}}. \tag{1.13.7}$$

The inverse transformations have the form

$$x_{\text{new}} = \frac{\left(1 + \frac{v}{c_1} + \frac{v}{c_2}\right) x_{\text{old}} - vt_{\text{old}}}{\sqrt{(1 + v/c_1)(1 + v/c_2)}},$$

$$t_{\text{new}} = \frac{t_{\text{old}} + \frac{v}{c_1 c_2} x_{\text{old}}}{\sqrt{(1 + v/c_1)(1 + v/c_2)}},$$

$$y_{\text{new}} = y_{\text{old}}, \qquad z_{\text{new}} = z_{\text{old}}. \tag{1.13.8}$$

Consequently, in the case of the metric (1.12.11) transformation formulas corresponding to transition between different inertial reference frames differ markedly from the Lorentz transformations. In the special case that $c_1 = -c_2 = c$, the transformations (1.13.7) and (1.13.8) change to the Lorentz transformations. It is also worth noting that if the direct and inverse Lorentz transformations are related rather simply by changing v for $-v$, in the general case there is no such relation between transformations (1.13.7) and (1.13.8).

Transformations (1.13.7) leave the expression (1.12.11) for the interval form-invariant and substitute for the Lorentz transformations in generalized inertial reference frames. In the case we have just considered the relative coordinate velocity **v** (the parameter of the transformation group) of generalized inertial frames was aligned along the x-axis. If the relative velocity of inertial reference frames has components along other axes as well, then the transformation formulas become markedly more complicated.

For instance, if the relative (physical) velocity of inertial frames with parallel and similarly directed space axes has the components $\mathbf{V} = [V^x, V^y, 0]$, then transformations that leave the metric (1.12.11) form-invariant and that correspond to the transition between different generalized inertial frames will be

$$
\begin{aligned}
x_{\text{old}} &= \\
&= \frac{2c}{c_2 - c_1}\Bigg[x_{\text{new}}\Bigg[\frac{c_2 - c_1}{2c}\left(\frac{V_y^2}{V^2} + \frac{V_x^2}{V^2\sqrt{1 - V^2/c^2}}\right) \\
&\quad - \frac{c_1 + c_2}{2c^2}\frac{V^x}{\sqrt{1 - V^2/c^2}}\Bigg] \\
&\quad + \frac{y_{\text{new}}}{\sqrt{-g_{11}}}\Bigg[\frac{(c_2 - c_1)V^xV^y}{2cV^2}\left(\frac{1}{\sqrt{1 - V^2/c^2}} - 1\right) \\
&\quad - \frac{c_1 + c_2}{2c^2}\frac{V^y}{\sqrt{1 - V^2/c^2}}\Bigg] \\
&\quad + t_{\text{new}}\Bigg[\frac{c_1^2 - c_2^2}{4c}\left(\frac{V_y^2}{V^2} + \frac{V_x^2}{V^2\sqrt{1 - V^2/c^2}}\right) \\
&\quad - \frac{c_1^2 - c_2^2}{4c\sqrt{1 - V^2/c^2}} - \frac{c_1c_2V^x}{c^2\sqrt{1 - V^2/c^2}}\Bigg]\Bigg], \\
y_{\text{old}} &= \sqrt{-g_{11}}\Bigg[x_{\text{new}}\frac{V^xV^y}{V^2}\left[\frac{1}{1 - V^2/c^2} - 1\right] \\
&\quad + \frac{y_{\text{new}}}{\sqrt{-g_{11}}}\left[\frac{V_x^2}{V^2} + \frac{V_y^2}{V^2\sqrt{1 - V^2/c^2}}\right] \\
&\quad + t_{\text{new}}\Bigg[\frac{c_1 - c_2}{2c}\frac{V^y}{\sqrt{1 - V^2/c^2}} \\
&\quad - \frac{c_1 + c_2}{2}\frac{V^xV^y}{V^2}\left(\frac{1}{\sqrt{1 - V^2/c^2}} - 1\right)\Bigg]\Bigg], \qquad (1.13.9) \\
t_{\text{old}} &= \frac{2c}{(c_1 - c_2)\sqrt{1 - V^2/c^2}}\Bigg[\frac{x_{\text{new}}V^x}{c^2} + \frac{y_{\text{new}}V^y}{c^2\sqrt{-g_{11}}} \\
&\quad + t_{\text{new}}\left[\frac{c_1 - c_2}{2c} - \frac{(c_1 + c_2)V^x}{2c^2}\right]\Bigg], \\
z_{\text{old}} &= z_{\text{new}}.
\end{aligned}
$$

At $c_1 = c_2 = c$ the transformations (1.13.9) go into corresponding Lorentz transformations

$$X_{\text{old}} = X_{\text{new}} \left[\frac{V_y^2}{V^2} + \frac{V_x^2}{V^2\sqrt{1 - V^2/c^2}} \right]$$

$$+ Y_{\text{new}} \frac{V^x V^y}{V^2} \left[\frac{1}{\sqrt{1 - V^2/c^2}} - 1 \right] + \frac{V^x T_{\text{new}}}{\sqrt{1 - V^2/c^2}},$$

$$Y_{\text{old}} = X_{\text{new}} \frac{V^x V^y}{V^2} \left[\frac{1}{\sqrt{1 - V^2/c^2}} - 1 \right]$$

$$+ Y_{\text{new}} \left[\frac{V_x^2}{V^2} + \frac{V_y^2}{V^2\sqrt{1 - V^2/c^2}} \right] + \frac{V^y T_{\text{new}}}{\sqrt{1 - V^2/c^2}},$$

$$T_{\text{old}} = \frac{X_{\text{new}} V^x + Y_{\text{new}} Y^y + c^2 T_{\text{new}}}{c^2\sqrt{1 - V^2/c^2}}, \quad Z_c = Z_{\text{new}}.$$

And so in the general case of arbitrary inertial reference frames there exists a three-parameter (whose parameters are three components of **V**) subgroup of coordinates transformations that leave the metric form-invariant and that correspond to a transition from one inertial frame to another.

1.14. Translation and Rotation Subgroup

We now show that in generalized inertial frames there exist subgroups of translations and three-dimensional rotations under which the metric (1.12.11) remains form-invariant.

We can easily see that with respect to the subgroup of translations of the time ($i = 0$) and space ($i = 1, 2, 3$) coordinates

$$x^i_{\text{old}} = x^i_{\text{new}} + a^i \tag{1.14.1}$$

by a constant vector a^i the metric (1.12.11) remains form-invariant. Indeed, from (1.14.1) it follows that

$$dx^i_{\text{old}} = dx^i_{\text{new}}.$$

Using this and considering that the components of the metric tensor in (1.12.11) have no explicit dependence on the coordinates, we can rewrite the metric in form (1.13.2), which proves that it is form-invariant under the transformation (1.14.1).

We notice further that the metric (1.12.11) in the yz-plane coincides with the metric of a Euclidean plane and so we conclude that the metric (1.12.11) also remains form-invariant under rotation of a three-dimen-

sional system of coordinates about the x-axis by an arbitrary constant angle.

Let us find further two transformations that describe in the same generalized inertial reference frame rotations about the y- and z-axes by arbitrary constant angles such that they leave the metric (1.12.11) form-invariant. A rotation about an arbitrary axis in a generalized inertial reference frame will be described by a corresponding product of rotations about x-, y- and z-axes taken in a definite order. We also note that we just have to find the transformation of rotation about one of the axes, say, the y-axis, since by a rotation about the x-axis we can always achieve a coincidence of any rotation axis in the xy-plane with the y-axis.

It is common knowledge [14] that the most general transformation or coordinates under which the reference frame does not change has the form

$$x'^{\alpha} = f^{\alpha}(x^{\beta}), \qquad x'^{0} = f^{0}(x^{\beta}, x^{0}).$$

The problem being symmetric, we write the required transformations of coordinated as

$$\begin{aligned} x_{\text{old}} &= F(x_{\text{new}} z_{\text{new}}), & z_{\text{old}} &= \Phi(x_{\text{new}}, z_{\text{new}}), \\ t_{\text{old}} &= \Psi(t_{\text{new}}, x_{\text{new}}, z_{\text{new}}), & y_{\text{old}} &= y_{\text{new}}. \end{aligned} \tag{1.14.2}$$

Since in this case the components of the metric tensor (1.12.11) have no explicit dependence on coordinates, the conditions for the metric (1.12.11) to remain form-invariant under the coordinate transformation (1.14.2) will, by the tensor law of the transformation of the metric tensor

$$g_{ik}^{\text{new}}(x_{\text{new}}) = \frac{\partial x_{\text{old}}^{l}}{\partial x_{\text{new}}^{i}} \frac{\partial x_{\text{old}}^{m}}{\partial x_{\text{new}}^{k}} \overset{c}{g}_{lm}(x_{\text{old}}(x_{\text{new}}))$$

take the form

$$\begin{aligned} g_{00} &= \Psi_t^2 g_{00}, \\ g_{01} &= c\Psi_t \Psi_x g_{00} + \Psi_t F_x g_{01}, \\ 0 &= c\Psi_t \Psi_z g_{00} + \Psi_t F_z g_{01}, \\ g_{11} &= c^2 \Psi_x^2 g_{00} + 2c\Psi_x F_x g_{01} + g_{11} F_x^2 - \Phi_x^2, \\ 0 &= c^2 \Psi_x \Psi_z g_{00} + c[\Psi_x F_z + \Psi_z F_x] g_{01} + g_{11} F_x F_z - \Phi_x \Phi_z, \\ -1 &= c^2 \Psi_z^2 g_{00} + 2c\Psi_z g_{01} + g_{11} F_z^2 - \Phi_z^2, \end{aligned} \tag{1.14.3}$$

where we have introduced the notation

$$\Psi_t = \frac{\partial \Psi}{\partial t_{\text{new}}}, \quad \Psi_x = \frac{\partial \Psi}{\partial x_{\text{new}}}, \quad \Psi_z = \frac{\partial \Psi}{\partial z_{\text{new}}},$$

$$F_x = \frac{\partial F}{\partial x_{\text{new}}}, \quad F_z = \frac{\partial F}{\partial z_{\text{new}}},$$

$$\Phi_x = \frac{\partial \Phi}{\partial x_{\text{new}}}, \quad \Phi_z = \frac{\partial \Phi}{\partial z_{\text{new}}}.$$

From the first of (1.14.3) it then follows that

$$\Psi = t_{\text{new}} + u(x_{\text{new}}, z_{\text{new}}).$$

From the second and third equations we have

$$u_z = -F_z \frac{g_{01}}{c g_{00}},$$

$$u_x = \frac{g_{01}}{c g_{00}} [1 - F_x]. \tag{1.14.4}$$

Substituting these into the fourth and sixth equations, we obtain

$$\Phi_z = \pm \sqrt{1 - F_z^2 \left(\frac{g_{01}^2}{g_{00}} - g_{11} \right)},$$

$$\Phi_x = \pm \sqrt{(1 - F_x^2) \left(\frac{g_{01}^2}{g_{00}} - g_{11} \right)}. \tag{1.14.5}$$

From (1.14.4) and (1.14.5) the remaining equation of the system (1.14.3) then becomes

$$F_x^2 - 1 + F_x^2 \left(\frac{g_{01}^2}{g_{00}} - g_{11} \right) = 0. \tag{1.14.6}$$

We will seek the solution of this nonlinear differential equation by the Lagrange – Charpy method.

Introducing the notation $P_1 = F_x$, $P_2 = F_z$ and compiling the characteristic system corresponding to the equation (1.14.6), we will obtain

$$\frac{dx_{\text{new}}}{2P_1} = \frac{dz_{\text{new}}}{2P_2 \left(\frac{g_{01}^2}{g_{00}} - g_{11} \right)} = -\frac{dF}{2} = \frac{dP_1}{0} = \frac{dP_2}{0}.$$

It follows that P_1 and P_2 are constants, i.e.

$$F_x = P_1 = \alpha = \text{const}, \quad F_z = P_2 = \beta = \text{const}. \tag{1.14.7}$$

Constant P_1 and P_2 are not independent, however; substituting (1.14.7) into (1.14.6) gives

$$1 - \alpha^2 = \beta^2 \left(\frac{g_{01}^2}{g_{00}} - g_{11} \right) = \beta^2 \kappa_{11}.$$

This relation will be valid if we assume that

$$\alpha = \pm \cos q, \qquad \beta = \mp \frac{\sin q}{\sqrt{\kappa_{11}}}.$$

From (1.14.7) we then find that

$$F = \pm x_{\text{new}} \cos q \mp \frac{z_{\text{new}} \sin q}{\sqrt{\kappa_1}} + x_0.$$

Since we are not interested in the inversion of the sign of the coordinate and in three-dimensional translations, we have

$$F = x_{\text{new}} \cos q - \frac{z_{\text{new}} \sin q}{\sqrt{\kappa_{11}}}. \tag{1.14.8}$$

Substituting (1.14.8) into (1.14.4) and (1.14.5) and solving them in a similar manner, we obtain

$$\Phi = x_{\text{new}} \sqrt{\kappa_{11}} \sin q + z_{\text{new}} \cos q,$$

$$u = x_{\text{new}} \frac{g_{01}}{c g_{00}} (1 - \cos q) + z_{\text{new}} \frac{g_{01} \sin q}{c g_{00} \sqrt{\kappa_{11}}}.$$

Transformations of rotation about the y-axis, such that the metric (1.12.11) remains form-invariant, will thus be

$$\begin{aligned}
t_{\text{old}} &= t_{\text{new}} + x_{\text{new}} \frac{g_{01}}{c g_{00}} (1 - \cos q) + z_{\text{new}} \frac{g_{01} \sin q}{c g_{00} \sqrt{\kappa_{11}}}, \\
x_{\text{old}} &= x_{\text{new}} \cos q - z_{\text{new}} \frac{\sin q}{\sqrt{\kappa_{11}}}, \\
z_{\text{old}} &= x_{\text{new}} \sqrt{\kappa_{11}} \sin q + z_{\text{new}} \cos q, \\
y_{\text{old}} &= y_{\text{new}}.
\end{aligned} \tag{1.14.9}$$

The inverse transformations are given by

$$t_{\text{new}} = t_{\text{old}} + x_{\text{old}} \frac{g_{01}}{c g_{00}} (1 - \cos q) - z_{\text{old}} \frac{g_{01} \sin q}{c g_{00} \sqrt{\kappa_{11}}},$$

$$x_{\text{new}} = x_{\text{old}} \cos q + z_{\text{old}} \frac{\sin q}{\sqrt{\kappa_{11}}},$$

$$z_{\text{new}} = - x_{\text{old}} \sqrt{\kappa_{11}} \sin q + z_{\text{old}} \cos q,$$

$$y_{\text{new}} = y_{\text{old}}. \tag{1.14.10}$$

As is seen from the expressions (1.14.9) and (1.14.10), when the coordinate system is rotated about the y-axis, this transformation also influences the coordinate time t, since the variables t and x are not orthogonal here.

We have thus found by direct calculation that in arbitrary permissible coordinates of a pseudo-Euclidean space-time there also exists a ten-parameter group of transformations under which the metric of this space-time remains form-invariant.

It is hardly a surprise, then, since the presence or absence of such a group is wholly dictated by the structure of space-time and is in no way conditioned by the choice of permissible coordinates. In any permissible coordinate system and with any arithmetization technique Minkowski's space-time always remains plane and its curvature tensor, which characterizes the internal structure of space-time, will always be zero.

1.15. Composition of Coordinate Velocities

Let us now establish the law for the composition of coordinate velocities under the transformations (1.13.8). Suppose, for instance, that in an "old" coordinate system the components of the coordinate velocity of a body have the form

$$v^x = dx/dt,$$
$$v^y = dy/dt,$$
$$v^z = dz/dt.$$

We now find the components of the coordinate velocity of the body in the "new" system, which moves relative to the "old" one with a coordinate velocity v. We will divide the differentials of the coordinates dx, dy, and dz by the differential dt. As a result,

we will have

$$
\begin{aligned}
v_{\text{new}}^{x} &= \frac{dx_{\text{new}}}{dt_{\text{new}}} = \frac{\left(1 + \dfrac{v}{c_1} + \dfrac{v}{c_2}\right) v_{\text{c}}^{x} - v}{1 + \dfrac{v v_{\text{c}}^{x}}{c_1 c_2}}, \\
v_{\text{new}}^{y} &= \frac{dy_{\text{new}}}{dt_{\text{new}}} = \frac{v_{\text{c}}^{y} \sqrt{\left(1 + \dfrac{v}{c_1}\right)\left(1 + \dfrac{v}{c_2}\right)}}{1 + \dfrac{v v_{\text{c}}^{x}}{c_1 c_2}}, \\
v_{\text{new}}^{z} &= \frac{dz_{\text{new}}}{dt_{\text{new}}} = \frac{v_{\text{c}}^{z} \sqrt{\left(1 + \dfrac{v}{c_1}\right)\left(1 + \dfrac{v}{c_2}\right)}}{1 + \dfrac{v v_{\text{c}}^{x}}{c_1 c_2}}.
\end{aligned}
\tag{1.15.1}
$$

We are thus led to conclude that the law of the composition of the coordinate velocities (1.15.1) in transition between different reference frames (1.13.8) differs markedly from that for the Lorentz transformations, although at $c_1 = -c_2 = c$ the laws coincide. We also note that the velocities c_1 and c_2 are limiting velocities of motion along the positive and negative x-axis, respectively. Indeed, examining the first of (1.15.1) for extremum when $c_1 > v > c_2$, we can see that v_{new}^{x} has a minimum at $v_{\text{old}}^{x} = c_2$ $(c_2 < 0)$ and a maximum at $v_{\text{new}}^{x} = c_1$. Moreover, it is of interest to note that on substituting into the first of (1.15.1) $v_{\text{old}}^{x} = c_1$ we will find that $v_{\text{new}}^{x} = c_1$ as well. If we assume that $v_{\text{old}}^{x} = c_2$, we will find $v_{\text{new}}^{x} = c_2$. Consequently, in any inertial system (1.15.8) the coordinate velocity of light along the positive x-axis is c_1, and in the opposite direction c_2, which was to be expected.

We now consider the law of composing physical velocities under the coordinate and time transformations (1.13.8). We will denote the physical velocity by V and the coordinate velocity by v.

As we have already mentioned, the components of the physical velocity are determined through the ratio of the physical time (1.10.12) and the physical distances (1.10.13). Using the metric (1.12.11) we

find that in our case

$$\kappa_{11} = g_{00}\frac{c^2(c_2 - c_1)^2}{4(c_1c_2)^2}, \qquad \kappa_{22} = 1, \qquad \kappa_{33} = 1,$$

$$g_{01} = -\frac{c}{2c_1c_2}(c_1 + c_2)g_{00}. \tag{1.15.2}$$

We can then write the expression

$$V^x = \frac{dl^x}{d\tau} = \frac{\sqrt{\kappa_{11}}dx}{dt\sqrt{g_{00}} + \dfrac{g_{01}dx}{c\sqrt{g_{00}}}} = \frac{\sqrt{\kappa_{11}}v^x}{\sqrt{g_{00}} + \dfrac{g_{01}v^x}{c\sqrt{g_{00}}}} \tag{1.15.3}$$

in the form

$$V^x = c\frac{(c_2 - c_1)v^x}{2c_1c_2 - (c_1 + c_2)v^x}. \tag{1.15.4}$$

Solving this for v^x gives

$$v^x = \frac{2c_1c_2V^x}{c(c_2 - c_1) + (c_1 + c_2)V^x}. \tag{1.15.5}$$

Similarly, from the expressions

$$\begin{aligned} V^y &= \frac{dl^y}{d\tau} = \frac{v^y}{\sqrt{g_{00}}\left[1 - \dfrac{(c_1 + c_2)}{2c_1c_2}v^x\right]}, \\ V^z &= \frac{dl^z}{d\tau} = \frac{v^z}{\sqrt{g_{00}}\left[1 - \dfrac{(c_1 + c_2)}{2c_1c_2}v^x\right]} \end{aligned} \tag{1.15.6}$$

we can find the remaining components of the coordinate velocity

$$\begin{aligned} v^y &= \frac{c(c_2 - c_1)\sqrt{g_{00}}V^y}{[(c_1 + c_2)V^x + c(c_2 - c_1)]}, \\ v^z &= \frac{c(c_2 - c_1)\sqrt{g_{00}}V^z}{[(c_1 + c_2)V^x + c(c_2 - c_1)]}. \end{aligned} \tag{1.15.7}$$

Suppose now that a body in the "old" reference frame has a physical velocity $\mathbf{V}_1$. We would like to find the physical velocity $\mathbf{V}_2$ in a "new" frame that travels along the x-axis with a velocity V relative to the "old" one.

It follows from the expressions (1.15.4) and (1.15.6) that the components of the physical velocity $\mathbf{V}_{(2)}$ are related to the components of the coordinate velocity $\mathbf{v}_{(2)}$ by

$$
\begin{aligned}
V^x_{(2)} &= \frac{c(c_2 - c_1)v^x_{(2)}}{2c_1c_2 - (c_1 + c_2)v^x_{(2)}},\\
V^y_{(2)} &= \frac{v^y_{(2)}}{\sqrt{g_{00}}\left[1 - \dfrac{c_1 + c_2}{2c_1c_2}\, v^x_{(2)}\right]},\\
V^z_{(2)} &= \frac{v^z_{(2)}}{\sqrt{g_{00}}\left[1 - \dfrac{c_1 + c_2}{2c_1c_2}\, v^x_{(2)}\right]}.
\end{aligned}
\tag{1.15.8}
$$

On the other hand, the components of the coordinate velocity $\mathbf{v}_{(2)}$ in the "new" reference frame are related to the components $\mathbf{v}_{(1)}$ of this velocity in the "old" system by (1.15.1). If now we substitute (1.15.1) into (1.15.8), take into consideration the relations (1.15.5) and (1.15.7) between the components of the coordinate velocity $\mathbf{v}_{(1)}$ and physical velocity $\mathbf{V}_{(1)}$, and recall that

$$
v = -\frac{dx_{\text{new}}}{dt_{\text{new}}} = -\frac{2c_1c_2V}{[(c_1 + c_2)V + c(c_2 - c_1)]},
\tag{1.15.9}
$$

we will obtain formulas for composing physical velocities

$$
\begin{aligned}
V^x_{(2)} &= \frac{V^x_{(1)} + V}{1 + \dfrac{VV^x_{(1)}}{c^2}},\\
V^y_{(2)} &= \frac{V^y_{(1)}\sqrt{1 - V^2/c^2}}{1 + \dfrac{VV^x_{(1)}}{c^2}},\\
V^z_{(2)} &= \frac{V^z_{(1)}\sqrt{1 - V^2/c^2}}{1 + \dfrac{VV^x_{(1)}}{c^2}}.
\end{aligned}
\tag{1.15.10}
$$

We thus find that physical velocities are composed here in the same manner as in the coordinate system where the metric is diagonal. Consequently, we see that, although a description can be given in arbitrary coordinates and it differs somewhat from the case of the diagonal metric,

for physical quantities we will at all times have expressions that coincide with that case.

Further, it follows from (1.15.4) that the physical velocity of light along the positive x-axis ($v^x = c_1$, $v^y = v^z = 0$) has an absolute value equal to the opposite velocity ($v^x = c_2$, $v^y = v^z = 0$) and coincides with the electrodynamic constant c.

We also note that although the coordinate velocities of bodies can have arbitrary magnitudes (but not higher, of course, than the coordinate velocities of light c_1 and c_2), the physical velocities can never be higher than c, i.e., $V < c$.

1.16. Examples of Generalized Inertial Reference Frames

Consider some examples of generalized inertial frames. As we have seen, to construct a generalized inertial reference frame we should change from Galilean coordinates (T, X, Y, Z) to new coordinates (t, x, y, z) related to the Galilean coordinates by the relations (1.12.2). Since (1.12.2) is a linear transformation, in the general case it describes the rotation of the x- and t-axes in the XT-plane. After the rotation the x-axis may end up nonorthogonal to the t-axis. And so rotations of either of the x- and t-axes in the XT-plane in an arbitrary manner through angles that need not be equal are generators of the general transformation (1.12.2), therefore it would be instructive to consider the transformations separately.

(1) Rotation of the x-axis without changing the orientation of the t-axis is described by a Galilean transformation

$$X = x - ut, \qquad T = t. \tag{1.16.1}$$

The components of the metric tensor can be derived from (1.12.4), if we take into account that in our case $p = a = 1$, $q = 0$, $b = -u$. We thus have

$$g_{00} = 1 - \frac{u^2}{c^2}, \quad g_{11} = g_{22} = g_{33} = -1, \qquad g_{01} = \frac{u}{c}. \tag{1.16.2}$$

The condition for the transformation (1.16.1) to be permissible will then be

$$g_{00} = 1 - \frac{u^2}{c^2} > 0.$$

Hence the limitation on the value of the parameter that appears in (1.16.1)

$$u^2 < c^2.$$

Using the expressions (1.16.2) for the components of the metric tensor, we then find from the relations (1.12.6) the values of the coordinate velocity of light along the x-axis and in the opposite direction

$$c_1 = c + u, \qquad c_2 = -c + u.$$

If for definiteness we put $u > 0$, we will then obtain by the condition $u^2 < c^2$

$$2c > c_1 \geqslant c, \quad 0 > c_2 \geqslant -c.$$

Coordinate transformations under which the metric tensor (1.16.2) remains form-invariant and which correspond to a transition from one inertial frame to another will then assume the form

$$x_{\text{new}} = \frac{\left(1 + \dfrac{uV}{c^2}\right) x_{\text{old}} + \left(1 - \dfrac{u^2}{c^2}\right) V t_{\text{old}}}{\sqrt{1 - V^2/c^2}},$$

$$t_{\text{new}} = \frac{\left(1 - \dfrac{uV}{c^2}\right) t_{\text{old}} + \dfrac{V x_{\text{c}}}{c^2}}{\sqrt{1 - V^2/c^2}},$$

where V is the physical relative velocity of the two frames under consideration.

(2) Rotation of the t-axis without changing the orientation of the x-axis in described by the transformation

$$X = x, \quad T = t + \frac{wx}{c^2}. \tag{1.16.3}$$

The components of the metric tensor will then be

$$g_{00} = 1, \; g_{01} = \frac{w}{c}, \; g_{11} = -\left(1 - \frac{w^2}{c^2}\right), \; g_{22} = g_{33} = -1. \tag{1.16.4}$$

The condition for the transformation (1.16.3) to be permissible yields the inequality

$$w^2 < c^2.$$

From (1.12.6), the velocity of light along the x-axis will be given by

$$c_1 = \frac{c^2}{c - w}, \qquad c_2 = -\frac{c^2}{c + w}.$$

If we assume that $w \geqslant 0$, then

$$c \leqslant c_1 < \infty, \quad -\frac{c}{2} > c_2 \geqslant -c.$$

The coordinate transformations that leave the metric tensor (1.16.4) form-invariant and that correspond to transition from one generalized inertial frame to another will then be

$$x_{\text{new}} = \frac{\left(1 + \frac{Vw}{c^2}\right) x_{\text{old}} + Vt_{\text{old}}}{\sqrt{1 - \frac{V^2}{c^2}}},$$

$$t_{\text{new}} = \frac{\left(1 - \frac{Vw}{c^2}\right) t_c + x_c \frac{V}{c^2}\left(1 - \frac{w^2}{c^2}\right)}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

We are thus led to conclude that in an arbitrary inertial reference frame Einstein's postulate of the constancy of the velocity of light in the forward and reverse directions does not hold, but nevertheless this in no way affects our possibility to describe physical processes in these reference frames.

1.17. Clock Synchronization at Different Points of Space

Consider clocks with the same rate of time flow located at different points in space. To bring their indications into correspondence with each other they have to be synchronized. Synchronization using light signals was first suggested by Poincaré [15] and was later used by Einstein [4].

The process occurs essentially as follows. An observer at a point A at a time $t = t_1$ (by his clock) sends to another point B a light signal. The clocks at points A and B will be synchronized, if the clock at B is started by the arrival of the light signal, the initial indication of the clock t' must take into account the time of the propagation t_{AB} of the light signal from A to B: $t' = t_1 + t_{AB}$. But the clock at A cannot measure the duration of the propagation of the light signal t_{AB} from A to B, since the beginning of the process (the sending of the signal) and its end (the reception of the signal) occur at different points in space.

We can use the clocks to measure only the duration of the process that begins and terminates at the same point, e.g., the total time taken

by the light signal to travel from A to B and back:

$$t_{ABA} = t_{AB} + t_{BA} = t_2 - t_1,$$

where the moments $t = t_1$ of sending the signal from point A to point B and of its return $t = t_2$ from B to A are timed according to the observer at A.

Assuming the velocity of light in the forward and reverse directions is the same ($t_{AB} = t_{BA}$), the clocks at points A and B will be synchronized if when the light signal arrives at B the clock there will show

$$t' = t_1 + \frac{1}{2}(t_2 - t_1).$$

Without this assumption we will have

$$t' = t_1 + \epsilon_{AB}(t_2 - t_1), \tag{1.17.1}$$

where ϵ_{AB} is a parameter characterizing the fact that the velocities of light in the opposite directions are unequal, i.e., $0 < \epsilon_{AB} < 1$.

Einstein [8] in 1917 admitted that the value $\epsilon = 1/2$ for clock synchronization is not obligatory, although in his works he used no other value.

Later on, in 1928, Reichenbach (see e.g. [10]) returned to this issue and looked at the possibility of synchronizing clocks for $\epsilon \neq 1/2$. Since in the Einstein – Reichenbach approach nothing predetermines the magnitude of the velocity of light in the forward and reverse directions, Reichenbach concluded that the choice of the quantity $0 < \epsilon < 1$ is not predetermined as well, and it is rather a convention. This suggested that such mechanical concepts as simultaneity, and mechanics itself for that matter, are conventional in nature.

These inferences are erroneous, however. As we have seen, the magnitude of the coordinate velocity of the light signal in various directions is no subject of convention, and it wholly predetermined by the choice of a coordinate system, i.e. of the metric of space-time. Therefore, the choice of the value of ϵ in (1.17.1) will be totally predetermined by the metric. On the other hand, if we take account of the metric, we construct physical quantities reflecting the behavior of space-time.

Indeed, consider, for instance, the synchronization of clocks aligned along the x-axis in an arbitrary inertial frame of reference with a metric given by (1.12.3). Without loss of generality, we will think that $x_A < x_B$; then, light will propagate from A to B with the velocity

$$c_1 = c\frac{g_{01} + \sqrt{g_{01}^2 - g_{00}g_{11}}}{-g_{11}} > 0, \tag{1.17.2}$$

In the opposite direction with the velocity

$$c_2 = c \frac{g_{01} - \sqrt{g_{01}^2 - g_{00}g_{11}}}{-g_{11}} < 0. \tag{1.17.3}$$

Suppose that an observer at a time t_1 sent from a point A to a point B a light signal that was reflected there and returned to A at t_2. The world lines of these points and the light signal are represented schematically in Fig. 3.

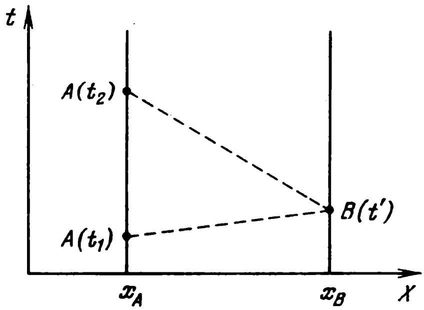

Fig. 3. The world lines of the observers and the light signal at clock synchronization

The signal travels from A to B with a velocity $dx/dt = c_2$. Since $c_1 =$ const, we can change in this expression from differentials to finite increments

$$l_{AB} = c_1 t_{AB},$$

where l_{AB} is the distance covered by the signal as it travelled from point A to point B.

Since $l_{AB} = l_{BA} = x_B - x_A$, we obtain from the above expressions

$$t_{AB} = -\frac{c_2}{c_1} t_{BA}. \tag{1.17.4}$$

The total time for the light signal to cover the distance ABA is

$$t_{ABA} = t_2 - t_1 = t_{AB} + t_{BA} = \frac{c_2 - c_1}{c_2} t_{AB} = \frac{c_1 - c_2}{c_1} t_{BA}.$$

Hence

$$t_{AB} = \frac{c_2}{c_2 - c_1}(t_2 - t_1),$$

$$t_{BA} = \frac{c_1}{c_1 - c_2}(t_2 - t_1). \tag{1.17.5}$$

By definition, the clocks at A and B will be synchronized, if at the moment the signal arrives at point B the clock there shows

$$t' = t_1 + t_{AB} = t_1 + \epsilon_{AB}(t_2 - t_1).$$

Substituting into this the first of (1.17.5), we will obtain

$$\epsilon_{AB} = \frac{c_2}{c_2 - c_1}, \tag{1.17.6}$$

Likewise, we can find that

$$\epsilon_{BA} = \frac{c_1}{c_1 - c_2}.$$

It is easily seen also that when the velocities of light in the opposite directions are equal, i.e., when $c_1 = -c_2$, we have

$$\epsilon_{AB} = \epsilon_{BA} = \frac{1}{2}.$$

We would now like to find ϵ for the special cases of inertial reference frames we have discussed in Section 1.16.

For the metric (1.16.2) the expressions (1.17.2) and (1.17.3) yield

$$c_1 = c + u, \qquad c_2 = -c + u.$$

Therefore, from (1.17.6),

$$\epsilon_{AB} = \frac{1}{2}\left(1 - \frac{u}{c}\right). \tag{1.17.7}$$

For the metric (1.16.4), we have from (1.17.2) and (1.17.3)

$$c_1 = \frac{c^2}{c - w}, \qquad c_2 = -\frac{c^2}{c + w}.$$

Then, by (1.17.6)

$$\epsilon_{AB} = \frac{1}{2}\left(1 - \frac{w}{c}\right). \tag{1.17.8}$$

It follows from (1.17.7) and (1.17.8) that the parameter ϵ_{AB} is the same (up to notation) when clocks are synchronized in the above two absolutely different inertial reference frames.

We next consider the synchronization of clocks aligned along some direction that is not coincident with the x-axis for the case of an arbitrary inertial reference frame (see Fig. 4). To be more specific, we will take points A and B lying in the xy-plane.

We will denote the slope of the straight line AB to the x-axis by k, i.e.,

$$\frac{dy}{dx} = k, \qquad |k| < \infty.$$

When a light signal travels along the direction AB, the interval (1.12.3) will be

$$ds^2 = c^2 g_{00}\, dt^2 + 2cg_{01}\, dx\, dt + (g_{11} - k^2)\, dx^2 = 0.$$

From this, the projections of the velocity of light on the x-axis will be

$$v_{1x} = c\,\frac{g_{01} + \sqrt{g_{01}^2 + (k^2 - g_{11})g_{00}}}{k^2 - g_{11}} > 0,$$

$$v_{2x} = c\,\frac{g_{01} - \sqrt{g_{01}^2 + (k^2 - g_{11})g_{00}}}{k^2 - g_{11}} < 0. \qquad (1.17.9)$$

Using similar arguments for clocks located along the x-axis, we obtain

$$\Delta x_{AB} = x_B - x_A = v_{1x} t_{AB} = -v_{2x} t_{BA}.$$

Hence

$$t_{AB} = -\frac{v_{2x}}{v_{1x}} t_{BA}.$$

The total time interval it takes the light signal to travel from point A

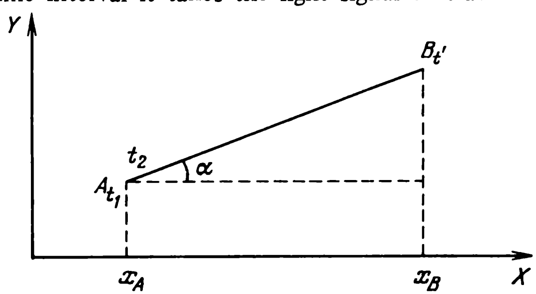

Fig. 4. Clock synchronization along an arbitrary direction

to point B and back will be

$$t_{ABA} = t_2 - t_1 = t_{AB} + t_{BA} = t_{AB}\left(1 - \frac{v_{1x}}{v_{2x}}\right).$$

From this relation we can express the time interval t_{AB} through the

total time interval

$$t_{AB} = \frac{v_{2x}}{v_{2x} - v_{1x}}(t_2 - t_1).$$

Using the definition

$$t' = t_1 + t_{AB} = t_1 + \epsilon_{AB}(t_2 - t_1),$$

we will find the expression for ϵ in this case

$$\epsilon_{AB} = \frac{v_{2x}}{v_{2x} - v_{1x}}. \tag{1.17.10}$$

And so the choice of the value of ϵ when synchronizing clocks at different points of space is not subject of convention or some principle of relativity theory, but is rather a particular manifestation of the specific choice of a coordinate system.

Physical events can, however, be described in any other permissible coordinates. One should only remember that the coordinate velocity of light, the value of ϵ at synchronization and also the relation of the coordinate quantities with physical quantities are completely determined by the choice of the coordinates, and hence by the metric tensor of space-time.

Meanwhile, Einstein in formulating the theory of relativity did not analyze this connection, he rather pursued the nonunique way of synchronization proposed earlier by Poincaré. As a result, as we know now, Einstein's postulate of the constancy of the velocity of light, as he formulated it, reflected only a particular selection of coordinate systems, for it relied on the concept of the coordinate velocity of light. At the time, he did not realize that one could define the velocity of light as a physical velocity only when there is a definition of length and time. We have also seen that the definition of time and length is bound to include the components of the metric tensor and the coordinate variables (see (1.4.12) and (1.4.13)). Einstein, as he noted in [16], understood this only in 1912. Galilean coordinate quantities coincide, or are connected in a simple manner, with the physical quantities $d\tau$ and dl^2:

$$d\tau = dT,$$

$$dl^2 = dX^2 + dY^2 + dZ^2.$$

1.18. Generalized Principle of Relativity

As we know now, Poincaré and Minkowski discovered that space-time, in which all physical processes occur, is a single whole and its geometry is pseudo-Euclidean. However this discovery divested the principle of relativity of its fundamental role and turned it into a particular consequence of the fact that all physical processes occur in space-time with pseudo-Euclidean geometry. This space-time retains its form-invariant metric under the transformations describing transition from one inertial reference frame to another, so making all inertial reference frames equivalent for the description of all physical events. Therefore, the fundamental role is now played by the geometry of space-time.

We will now see that the statement that all physical processes occur in space-time with pseudo-Euclidean geometry is much richer than the content of the principle of relativity, since this statement enables us to formulate the generalized principle of relativity that is valid not only in inertial but also in noninertial reference frames. It is to be noted in this connection that in the literature one often comes across the statement that the special theory of relativity is confined to description of phenomena in inertial frames of reference, where as to describe things in noninertial frames is the task of the general theory of relativity.

These statements are erroneous. If follows from Poincaré's and Minkowski's fundamental discoveries (that the geometry of space-time where all physical processes occur is pseudo-Euclidean) that to describe physical phenomena we can make use of any class of permissible frames of reference, inertial as well as noninertial. Therefore, within the framework of the special relativity theory it is possible to give a complete description of physical phenomena even in noninertial reference frames.

Since a transition between various reference frames does not change the geometry of space-time and it remains pseudo-Euclidean for any frame, inertial and noninertial alike, there exists a ten-parameter group of coordinate transformations under which the metric tensor remains form-invariant. In other words, in pseudo-Euclidean space-time for any reference frame we can indicate an infinite set of other frames such that transformations between them leave the metric form-invariant.

This implies that in pseudo-Euclidean space-time holds the generalized principle of relativity, which can be formulated as follows: in any physical frame (inertial or noninertial) we can always indicate an infinite set of other reference frames in which all physical events happen similarly as in the original reference frame. So we have no, and cannot have, experimental means to tell experimentally in which of the infinite

set of the frames we really are. If then we define for an arbitrary permissible reference frame coordinate transformations under which the metric remains form-invariant, we thereby find the entire infinite set of reference frames that are physically equivalent to the original one.

It should be stressed that any physical process permits to determine whether we are in an inertial or noninertial reference frame. Still no physical experiment can give an answer to the question: in which of the infinite set of frames with a form-invariant metric we reside? The discovery of the pseudo-Euclidean geometry of space-time enabled physical laws to be formulated both in inertial and in noninertial reference frames, thus disproving the erroneous statement concerning the inapplicability of special relativity to accelerated frames.

We will now take an example of a relativistically accelerated reference frame to illustrate how we can find all the sets of reference frames whose metric is form-invariant with the metric of the original frame.

1.19. Relativistic Uniformly Accelerated Motion

To form a relativistic uniformly accelerated reference frame we will have, above all, to make it clear what relativistic motion should be regarded as uniformly accelerated.

It is common knowledge that in classical mechanics uniformly accelerated motion of a material point is a motion under the action of a force constant in magnitude and direction, i.e.,

$$f^{\alpha} = \text{const.} \tag{1.19.1}$$

If we translate this definition to the relativistic case, it is only natural to associate the term "relativistic uniformly accelerated" only with a motion that occurs under the action of the force (1.19.1) such that it obeys the equations of relativistic mechanics

$$m_0 c \frac{du^i}{ds} = F^i, \tag{1.19.2}$$

where $u^i = dx^i/dx$ is the four-vector of the velocity of the particle

$$u^i = \left[\frac{1}{\sqrt{1 - v^2/c^2}}, \frac{v^{\alpha}}{c\sqrt{1 - v^2/c^2}} \right], \tag{1.19.3}$$

F^i is the four-vector of the force

$$F^i = \left[\frac{\mathbf{fv}}{c^2\sqrt{1 - v^2/c^2}}, \frac{f^{\alpha}}{c\sqrt{1 - v^2/c^2}} \right], \tag{1.19.4}$$

f^{α} is the conventional three-dimensional force.

In Galilean coordinates the interval for a moving particle, as usual, has the form

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{1.19.5}$$

Considering that for a particle that moves according to the law

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

the differentials dx, dy, dz are not independent, but are related to the differential dt by

$$dx = v^x dt, \quad dy = v^y dt, \quad dz = v^z dt,$$

the interval (1.19.5) becomes

$$ds^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right). \tag{1.19.6}$$

Note that the quantity $d\tau = dt\sqrt{1 - v^2/c^2}$ is known as the proper time of the moving particle.

The equations of motion (1.19.2), using (1.19.3), (1.19.4) and (1.19.6), become

$$\begin{aligned} m_0 \frac{d}{dt} \frac{1}{\sqrt{1 - v^2/c^2}} &= \frac{\mathbf{f}\mathbf{v}}{c^2}, \\ m_0 \frac{d}{dt} \frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} &= \mathbf{f}. \end{aligned} \tag{1.19.7}$$

These equations are due to Poincaré [3], although it is widely believed that it was Planck [17] who was the first to derive them. It is obvious that for small velocities ($v/c \ll 1$) the equations (1.19.7) change to the conventional equations of Newton

$$\frac{d}{dt} \frac{m_0 v^2}{2} = \mathbf{f}\mathbf{v}, \qquad m_0 \frac{d\mathbf{v}}{dt} = \mathbf{f}.$$

We now show that the first of (1.19.7) is a consequence of the others, and so it can be discarded. To this end, we write the three-dimensional equations of motion (1.19.7) in the form

$$\frac{m_0}{\sqrt{1 - v^2/c^2}} \frac{d\mathbf{v}}{dt} + \frac{m_0 \mathbf{v}}{(1 - v^2/c^2)^{3/2}} \left(\frac{\mathbf{v}}{c^2} \frac{d\mathbf{v}}{dt}\right) = \mathbf{f}. \tag{1.19.8}$$

Multiplying the equation (1.19.8) scalarly by v/c^2, we will find

$$\frac{m_0}{(1 - v^2/c^2)^{3/2}}\left(\frac{\mathbf{v}}{c^2}\,\frac{d\mathbf{v}}{dt}\right) = \frac{\mathbf{v}}{c^2}\,\mathbf{f}. \tag{1.19.9}$$

Rearranging the left-hand side, we will obtain the first of (1.19.7)

$$m_0\,\frac{d}{dt}\,\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{m_0}{(1 - v^2/c^2)^{3/2}}\left(\frac{\mathbf{v}}{c^2}\,\frac{d\mathbf{v}}{dt}\right) = \mathbf{f}\frac{\mathbf{v}}{c^2}.$$

We also note that by virtue of (1.19.9) the relativistic equations of motion (1.19.8) can be written in quasi-classical form

$$m_0\,\frac{d\mathbf{v}}{dt} = \left[\mathbf{f} - \frac{\mathbf{v}}{c^2}(\mathbf{vf})\right]\sqrt{1 - v^2/c^2}.$$

Therefore, relativistic uniformly accelerated motion must obey the equation

$$\frac{d}{dt}\,\frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \frac{\mathbf{f}}{m_0} = \mathbf{w} = \text{const.} \tag{1.19.10}$$

Let us now find the law governing the time variation of the coordinates of a particle moving relativistically and with uniform acceleration. Integrating (1.19.10) with respect to time gives

$$\frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \mathbf{w}t + \mathbf{v}_0, \quad \mathbf{v}_0 = \frac{\mathbf{v}(0)}{\sqrt{1 - v^2(0)/c^2}}. \tag{1.19.11}$$

We now divide this by c, square both sides and add to both sides a unity. As a result, we will have

$$\frac{1}{1 - v^2/c^2} = 1 + \frac{(\mathbf{w}t + \mathbf{v}_0)}{c^2}. \tag{1.19.12}$$

From (1.19.11) and (1.19.12) we will then have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\mathbf{w}t + \mathbf{v}_0}{\sqrt{1 + \dfrac{(\mathbf{w}t + \mathbf{v}_0)^2}{c^2}}}.$$

Integrating this differential equation, we will obtain the law of rela-

tivistic uniformly accelerated motion

$$\mathbf{r} = \mathbf{r}_0 + \frac{\mathbf{w}c^2}{w^2}\left[\sqrt{1 + \frac{(\mathbf{w}t + \mathbf{v}_0)^2}{c^2}} - 1\right]$$

$$+ \frac{c}{w}\left(\mathbf{v}_0 - \mathbf{w}\frac{(\mathbf{v}_0\mathbf{w})}{w^2}\right)\ln\left[\frac{wt}{c} + \frac{\mathbf{v}_0\mathbf{w}}{cw} + \sqrt{1 + \frac{(\mathbf{w}t + \mathbf{v}_0)^2}{c^2}}\right]. \tag{1.19.13}$$

The proper time of the particle will then change by the law

$$\tau = t_0 + \frac{c}{w}\ln\left[\frac{wt}{c} + \frac{\mathbf{v}_0\mathbf{w}}{cw} + \sqrt{1 + \frac{(\mathbf{w}t + \mathbf{v}_0)^2}{c^2}}\right]^2. \tag{1.19.14}$$

1.20. Group of Relativistic Uniformly Accelerated Frames

Suppose that an inertial relativistic and a uniformly accelerated reference frames have their axes aligned, and the latter one moves without an initial velocity (i.e., $\mathbf{v}_0 = 0$) along the x-axis of the inertial frame. Then, if we take the origins to coincide at $t = 0$, we will obtain by (19.13) the law of motion of the origin of the relativistic uniformly accelerated reference frame

$$x_0 = \frac{c^2}{w}\left[\sqrt{1 + \frac{w^2t^2}{c^2}} - 1\right].$$

Therefore, formulas of coordinate transformations in passing to the relativistic uniformly accelerated frame (x, t) will have the form

$$x = X - x_0 = X - \frac{c^2}{w}\left[\sqrt{1 + \frac{w^2T^2}{c^2}} - 1\right].$$

The time transformation can be defined arbitrarily. Consider two cases of special interest:

(a) Time remains the same in both frames, i.e.,

$t = T$;

(b) We take time to be the proper time of some point (e.g., of the origin) of the accelerated frame

$$t = \frac{c}{w}\operatorname{Arsinh}\frac{wT}{c} = \frac{c}{w}\ln\left[\frac{wT}{c} + \sqrt{1 + \frac{w^2T^2}{c^2}}\right].$$

For (a) we have

$$x = X - \frac{c^2}{w}\left[\sqrt{1 + \frac{w^2 T^2}{c^2}} - 1\right], \qquad t = T. \tag{1.20.1}$$

Notice that without acceleration ($w = 0$) this transformation turns into the identical transformation

$$x = X, \quad t = T.$$

Under the transformation (1.20.1) the metric of pseudo-Euclidean space-time becomes

$$ds^2 = \frac{c^2 dt^2}{1 + \frac{w^2 t^2}{c^2}} - \frac{2wt\,dt\,dx}{\sqrt{1 + \frac{w^2 t^2}{c^2}}} - dx^2 - dy^2 - dz^2. \tag{1.20.2}$$

We will now find out which coordinate transformations leave the metric (1.20.2) form-invariant. It is well known that in pseudo-Euclidean space-time there exists a ten-parameter group of infinitesimal motions for which the metric of that space-time remains form-invariant. The presence of the ten Killing vectors guarantees that there are ten conservation laws in any reference frame of pseudo-Euclidean space-time.

In this case, however, we will have to examine a group of finite coordinate transformations that leave the metric (1.20.2) form-invariant, but not of infinitesimal motions, characterized by the Killing vectors.

It is evident that the transformation of translation of the space coordinates

$$x'^{\alpha} = x^{\alpha} + a^{\alpha}$$

by the constant vector a^{α} leave the metric (1.20.2) form-invariant.

Indeed, since the metric tensor γ_{ni} is independent of space coordinates, and the factors $\partial x'^{\alpha}/\partial x^{\beta}$ form the three-dimensional Kronecker symbol

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta},$$

it follows from the tensor law of transformation of the components of the metric tensor

$$\gamma'^{np}(x') = \frac{\partial x'^{n}}{\partial x^{l}} \frac{\partial x'^{p}}{\partial x^{m}} \gamma^{lm}(x(x'))$$

that the form-invariance condition for the metric holds:

$$\gamma'^{np}(x') = \gamma^{np}(x').$$

It is easily seen also that the metric (1.20.2) is form-invariant event under the transformation of rotation about the x-axis

$$y' = y\cos\alpha - z\sin\alpha,$$
$$z' = y\sin\alpha + z\cos\alpha.$$

Let us now consider the coordinate transformations that correspond to transition between different noninertial reference frames and that leave the metric (1.20.2) form-invariant. For the sake of simplicity, we will first consider the case of the relative motion of the noninertial frame along the x-axis.

The problem being symmetrical, the required coordinate transformations can be written as

$$\begin{aligned} t_{\text{old}} &= \Psi(t_{\text{new}}, x_{\text{new}}), \\ x_{\text{old}} &= cf(t_{\text{new}}, x_{\text{new}}), \\ y_{\text{old}} &= y_{\text{new}}, \quad z_{\text{old}} = z_{\text{new}}. \end{aligned} \tag{1.20.3}$$

Under coordinate transformations the metric tensor of space-time transforms according to the law

$$\gamma_{ik}^{\text{new}}(x_{\text{new}}) = \frac{\partial x_{\text{old}}^{l}}{\partial x_{\text{new}}^{i}} \frac{\partial x_{\text{old}}^{m}}{\partial x_{\text{new}}^{k}} \gamma_{ml}^{\text{old}}(x_{\text{old}}(x_{\text{new}})), \tag{1.20.4}$$

where $\gamma_{ik}^{\text{new}}(x)$ and $\gamma_{ml}^{\text{old}}(x)$ are respectively the new and old functional forms of the metric tensor. The form-invariance condition for the metric requires that the functional form of the metric tensor under the coordinate transformation would remain unaltered

$$\gamma_{ik}^{\text{new}}(x) = \gamma_{ik}^{\text{old}}(x). \tag{1.20.5}$$

In the case under consideration the condition (1.20.5) requires that the metric tensor would have the form

$$\begin{aligned} \gamma_{00}^{\text{new}} &= \frac{1}{1 + \dfrac{w^2 t_{\text{new}}^2}{c^2}}, \quad \gamma_{01}^{\text{new}} = -\frac{w t_{\text{new}}}{c\sqrt{1 + \dfrac{w^2 t_{\text{new}}^2}{c^2}}}, \\ \gamma_{00}^{\text{old}} &= \frac{1}{1 + \dfrac{w^2 \Psi^2}{c^2}}, \quad \gamma_{01}^{\text{old}} = -\frac{w\Psi}{c\sqrt{1 + \dfrac{w^2\Psi^2}{c^2}}}, \end{aligned} \tag{1.20.6}$$

$$\gamma_{11}^{\text{old}} = \gamma_{11}^{\text{new}} = -1, \quad \gamma_{22}^{\text{new}} = \gamma_{22}^{\text{old}} = -1, \quad \gamma_{33}^{\text{new}} = \gamma_{33}^{\text{old}} = -1.$$

Because of the tensor nature of the transformations of the tensor (1.20.4) the form-invariance condition (1.20.5) under the transforma-

tion (1.20.3) can be written otherwise:

$$\gamma_{00}^{\text{new}} = \Psi_t^2 \gamma_{00}^{\text{old}} + 2\Psi_t f_t \gamma_{01}^{\text{old}} - f_t^2,$$

$$\gamma_{01}^{\text{new}} = c\Psi_t \Psi_x \gamma_{00}^{\text{old}} + c\gamma_{01}^{\text{old}}[\Psi_t f_x + \Psi_x f_t] - cf_t f_x, \qquad (1.20.7)$$

$$-1 = \Psi_x^2 \gamma_{00}^{\text{old}} c^2 + 2\Psi_x f_x \gamma_{01}^{\text{old}} c^2 - f_x^2 c^2,$$

where

$$\Psi_t = \frac{\partial \Psi}{\partial t}, \quad \Psi_x = \frac{\partial \Psi}{\partial x}, \quad f_t = \frac{\partial f}{\partial t}, \quad f_x = \frac{\partial f}{\partial x}.$$

From the first and the last of the equations of the system of nonlinear partial differential equations (1.20.7) we have

$$f_t = \Psi_t \gamma_{01}^{\text{old}} + a\sqrt{\Psi_t^2[(\gamma_{01}^{\text{old}})^2 + \gamma_{00}^{\text{old}}] - \gamma_{00}^{\text{new}}}, \qquad (1.20.8)$$

$$f_x = \Psi_x \gamma_{01}^{\text{old}} + b\sqrt{\Psi_x^2[(\gamma_{01}^{\text{old}})^2 + \gamma_{00}^{\text{old}}] + \frac{1}{c^2}},$$

where a and b are the sign functions

$$a = \pm 1, \quad b = \pm 1.$$

Substituting (1.20.8) into the second equation of (1.20.7) gives

$$[\Psi_t^2 + 2c\gamma_{01}^{\text{new}} \Psi_t \Psi_x - \gamma_{00}^{\text{new}} c^2 \Psi_x^2][(\gamma_{01}^{\text{old}})^2 + \gamma_{00}^{\text{old}}]$$

$$- \gamma_{00}^{\text{new}} - (\gamma_{01}^{\text{new}})^2 = 0. \qquad (1.20.9)$$

From (1.20.6), we write (1.20.9) as

$$\Psi_t^2 - \frac{c^2 \Psi_x^2}{1 + \dfrac{w^2 t^2}{c^2}} - \frac{2wt\Psi_t \Psi_x}{\sqrt{1 + \dfrac{w^2 t^2}{c^2}}} - 1 = 0. \qquad (1.20.10)$$

The system (1.20.7) has thus been reduced to one nonlinear partial differential equation (1.20.10). Solving this we can readily obtain the solutions of the equations (1.20.8).

According to the general theory of nonlinear partial differential equations of the first order (the Lagrange –Charpy method) equations

(1.20.10) have the following characteristic system ($P_1 = \Psi_t$, $P_2 = \Psi_x$):

$$\frac{dt}{2\left[P_1 - \dfrac{wtP_2}{\sqrt{1 + \dfrac{w^2t^2}{c^2}}}\right]}$$

$$= -\frac{dx}{2\left[\dfrac{P_2c^2}{1 + \dfrac{w^2t^2}{c^2}} + \dfrac{wtP_1}{\sqrt{1 + \dfrac{w^2t^2}{c^2}}}\right]}$$

$$= \frac{d\Psi}{2} \tag{1.20.11}$$

$$= \frac{dP_1}{\dfrac{2wP_2}{\left[1 + \dfrac{w^2t^2}{c^2}\right]^{3/2}}\left[P_1 - \dfrac{wtP_2}{\sqrt{1 + \dfrac{w^2t^2}{c^2}}}\right]} = \frac{dP_2}{0}.$$

Combining the last term with the first two terms yields

$$P_2 = \frac{\dot{\Phi}_0}{c}, \quad \Phi_0 = \text{const.} \tag{1.20.12}$$

Dividing the penultimate term of the characteristic system (1.20.11) by the first term and taking expression (1.20.11) into account, we arrive at

$$\frac{dP_1}{dt} = \frac{w\Phi_0}{c}\left[1 + \frac{w^2t^2}{c^2}\right]^{-3/2}$$

Integrating this ordinary differential equation gives

$$P_1 = \Phi_1 + \frac{\Phi_0 wt}{c\sqrt{1 + \dfrac{w^2t^2}{c^2}}}, \quad \Phi_1 = \text{const.} \tag{1.20.13}$$

It follows thus from (1.20.12) and (1.20.13) that

$$\Psi_t = \Phi_1 + \frac{\Phi_0 wt}{c\sqrt{1 + \dfrac{w^2 t^2}{c^2}}},$$
$$\Psi_x = \frac{\Phi_0}{c}. \tag{1.20.14}$$

Substituting these expressions into (1.20.10), we find that Φ_0 is related to Φ_1 by

$$\Phi_1^2 = 1 + \Phi_0^2. \tag{1.20.15}$$

Integrating the system of linear partial differential equations of the first order (1.20.14), we will have

$$\Psi = \frac{\Phi_0}{c} x + \Phi_1 t + \frac{c}{w}\Phi_0 \sqrt{1 + \frac{w^2 t^2}{c^2}} + \tilde{\Psi}_0,$$
$$\tilde{\Psi}_0 = \text{const.} \tag{1.20.16}$$

Substituting (1.20.14) into (1.20.8) gives

$$f_t = -\frac{c}{w}\frac{\partial}{\partial t}\sqrt{1 + \frac{w^2\Psi^2}{c^2}} + a\left[\Phi_0 + \frac{\Phi_1 wt}{c\sqrt{1 + \dfrac{w^2 t^2}{c^2}}}\right],$$

$$f_x = -\frac{c}{w}\frac{\partial}{\partial x}\sqrt{1 + \frac{w^2\Psi^2}{c^2}} + \frac{b\Phi_1}{c}.$$

Integrating this system, we will get

$$f = -\frac{c}{w}\sqrt{1 + \frac{\omega^2\Psi^2}{c^2}} + b\Phi_1\frac{x}{c} + a\Phi_0 t$$
$$+ \frac{c}{w}\Phi_1\sqrt{1\ 1 + \frac{w^2 t^2}{c^2}} + \tilde{f}_0.$$

The coordinate transformations under which the metric (1.20.2) remains

form-invariant, will have the form

$$t_{\text{old}} = \frac{\Phi_0}{c} x_{\text{new}} + \Phi_1 t_{\text{new}} + \frac{c}{w} \Phi_0 \sqrt{1 + \frac{w^2 t_{\text{new}}^2}{c^2}} + \tilde{\Psi}_0,$$

$$x_{\text{old}} = b\Phi_1 x_{\text{new}} + ca\Phi_0 t_{\text{new}} + \frac{c^2 \Phi_1}{w} \sqrt{1 + \frac{w^2 t_{\text{new}}^2}{c^2}} \tag{1.20.17}$$

$$\frac{c^2}{w} \sqrt{1 + \frac{w^2}{c^2} \left[\frac{\Phi_0}{c} x_{\text{new}} + \Phi_1 t_{\text{new}} + \frac{c}{w} \Phi_0 \sqrt{1 + \frac{w^2 t_{\text{new}}^2}{c^2}} + \tilde{\Psi}_0 \right]^2}$$

$$+ \tilde{f}_0 c.$$

For these transformations to be meaningful at $w = 0$ as well, we will have to redefine the integration constants

$$\tilde{\Psi}_0 = \Psi_0 \frac{c}{w} \Phi_0, \quad \tilde{f}_0 = \frac{1}{c} f_0 - \frac{c}{w} \Phi_1 + \frac{c}{w}.$$

We will then have

$$t_{\text{old}} = \frac{\Phi_0}{c} x_{\text{new}} + \Phi_1 t_{\text{new}} + \frac{c}{w} \Phi_0 \left[\sqrt{1 + \frac{w^2 t_{\text{new}}^2}{c^2}} - 1 \right] + \Psi_0,$$

$$x_{\text{old}} = b\Phi_1 x_{\text{new}} + ca\Phi_0 t_{\text{new}}$$

$$+ \frac{c^2}{w} \Phi_1 \left[\sqrt{1 + \frac{w^2 t_{\text{new}}^2}{c^2}} - 1 \right]$$

$$- \frac{c^2}{w} \left[\left\{ 1 + \frac{w^2}{c^2} \left[\frac{\Phi_0}{c} x_{\text{new}} + \Phi_1 t_{\text{new}} \right. \right. \right. \tag{1.20.18}$$

$$\left. \left. \left. + \frac{c}{w} \Phi_0 \left(\sqrt{1 + \frac{w^2 t_{\text{new}}^2}{c^2}} - 1 \right) + \Psi_0 \right]^2 \right\}^{1/2} - 1 \right] + f_0,$$

where $\Phi_1^2 = 1 + \Phi_0^2$. These transformations contain three arbitrary parameters Φ_0, Ψ_0, f_0 (the parameter w in the expressions (1.20.18) is given by the metric of the space-time (1.20.2) and is, therefore, no arbitrary parameter of the transformation) and two sign functions $a = \pm 1$ and $b = \pm 1$. It is quite obvious that the sign functions describe the inversion operations for the coordinates x and t. Since later in the book we are only going to be interested in the proper group without inversion, we will put $a = b = 1$.

The transformations (1.20.18) are thus a three-parameter group of coordinate transformations, such that the metric (1.20.2) remains form-invariant. We will now clarify the meaning of the group parameters. The parameter f_0 describes the translation of the coordinates x we have already discussed above. It is easily seen that the parameter Ψ_0 describes the time translation transformation.

Indeed, putting $\Phi_1 = 1$, $\Phi_0 = 0$, we will have

$$t_{\text{old}} = t_{\text{new}} + \Psi_0,$$

$$x_{\text{old}} = x_{\text{new}} + \frac{c^2}{w}\left[\sqrt{1 + \frac{w^2 t_{\text{new}}^2}{c^2}} - 1\right] - \frac{c^2}{w}\left[\sqrt{1 + \frac{w^2 (t_{\text{new}} + \Psi_0)^2}{c^2}} - 1\right] + f_0.$$

Unlike Poincaré's group, to provide form-invariance in a relativistic uniformly accelerated reference frame time translation requires that coordinates be changed as well.

Parameter Φ_0 describes the motion of one noninertial reference frame relative to another one. To derive the expression for this parameter in terms of the physical velocity V of the relative motion, we will make use of the following circumstance: in an infinitesimal neighborhood of the initial moment for both reference frames, i.e., when the conditions

$$\frac{w^2 t_c^2}{c^2} = \frac{w^2}{c^2}\left[\frac{\Phi_0}{c} x_{\text{new}} + \Phi_1 t_{\text{new}} + \frac{c}{w}\Phi_0\left(\sqrt{1 + \frac{w^2 t_{\text{new}}^2}{c^2}} - 1\right)\right]^2 \ll 1,$$

$$\frac{w^2 t_{\text{new}}^2}{c^2} \ll 1, \quad \Psi_0 = f_0 = 0 \tag{1.20.19}$$

are met, the transformations (1.20.18) must be Lorentz transformations between instantaneously comoving inertial reference frames.

If then we take account of the estimates (1.20.19), we will obtain by (1.20.18),

$$t_{\text{old}} = \frac{\Phi_0}{c} x_{\text{new}} + \Phi_1 t_{\text{new}}, \qquad x_{\text{old}} = \Phi_1 x_{\text{new}} + c\Phi_0 t_{\text{new}}.$$

Hence

$$\Phi_0 = \frac{V}{c\sqrt{1 - V^2/c^2}}, \qquad \Phi_1 = \frac{1}{\sqrt{1 - V^2/c^2}},$$

which is in agreement with relation (1.20.15).

The transformations under which the metric (1.20.2) remains form-invariant and which describe transition between relativistic uniformly accelerated frames will thus be

$$t_{\text{old}} = \frac{t_{\text{new}} + \dfrac{Vx_{\text{new}}}{c^2} + \dfrac{V}{w}\left(\sqrt{1 + \dfrac{w^2 t_{\text{new}}^2}{c^2}}\right)}{\sqrt{1 - V^2/c^2}} + \Psi_0,$$

$$x_{\text{old}} = \frac{x_{\text{new}} + Vt_{\text{new}} + \dfrac{c^2}{w}\left(\sqrt{1 + \dfrac{w^2 t_{\text{new}}^2}{c^2}} - 1\right)}{\sqrt{1 - V^2/c^2}}$$

$$- \frac{c^2}{w}\left[\left\{1 + \frac{w^2}{c^2 - V^2}\left[t_{\text{new}}\right.\right.\right.$$

$$\left.\left.\left. + \frac{Vx_{\text{new}}}{c^2} + \frac{V}{w}\left(\sqrt{1 + \frac{w^2 t_{\text{new}}^2}{c^2}} - 1\right) + \Psi_0\sqrt{1 - \frac{V^2}{c^2}}\right]^2\right\}^{1/2} - 1\right] + f_0.$$

(1.20.20)

The inverse transformations have the form

$$t_{\text{new}} = \frac{t_{\text{old}} - \dfrac{Vx_{\text{old}}}{c^2} - \dfrac{V}{w}\left(\sqrt{1 + \dfrac{w^2 t_c^2}{c^2}} - 1\right) - \Psi_0 + \dfrac{Vf_0}{c^2}}{\sqrt{1 - V^2/c^2}},$$

$$x_{\text{new}} = \frac{x_{\text{old}} - Vt_{\text{old}} + \dfrac{c^2}{w}\left(\sqrt{1 + \dfrac{w^2 t_{\text{old}}^2}{c^2}} - 1\right)}{\sqrt{1 - V^2/c^2}} \qquad (1.20.21)$$

$$- \frac{c^2}{w}\left[\left\{1 + \frac{w^2}{c^2 - V^2}\left[t_{\text{old}} - \frac{Vx_{\text{old}}}{c^2} - \frac{V}{w}\left(\sqrt{1 + \frac{w^2 t_{\text{old}}^2}{c^2}} - 1\right)\right.\right.\right.$$

$$\left.\left.\left. \Psi_0\sqrt{1 - \frac{V^2}{c^2}}\right]^2\right\}^{1/2} - 1\right] - \frac{f_0 - V\Phi_0}{\sqrt{1 - V^2/c^2}}.$$

Accordingly, in this case at $\Psi_0 = 0$ and $f_0 = 0$, just as in the case of Lorentz transformations, the expressions for the forward and reverse transformations can be derived from each other by changing the sign of the relative velocity $V \to -V$ and changing $t_{\text{new}} \leftrightarrow t_{\text{old}}$ and $x_{\text{new}} \leftrightarrow x_{\text{old}}$.

For the second case the formulas transformations from an inertial frame to a relativistic uniformly accelerated frame have the form

$$x = X - \frac{c^2}{w}\left[\sqrt{1 + \frac{w^2 T^2}{c^2}} - 1\right],$$
$$t = \frac{c}{w}\,\mathrm{Ar\,sin\,h}\,\frac{wT}{c}. \tag{1.20.22}$$

In the relativistic frame given by (1.20.22) the metric will be

$$ds^2 = c^2 dt^2 - 2\sin\mathrm{h}\frac{wt}{c}\,dxcdt - dx^2 - dy^2 - dz^2. \tag{1.20.23}$$

We will find the coordinate transformations of the form (1.20.3), for which the metric (1.20.23) remains form-invariant. In the case under consideration the nonzero components of the metric tensor of space-time in the old and new coordinate systems must be

$$\gamma_{01}^{\text{new}} = -\sin\mathrm{h}\frac{wt}{c}, \qquad \gamma_{01}^{\text{old}} = -\sin\mathrm{h}\frac{w\Psi}{c},$$

$$\gamma_{11}^{\text{new}} = \gamma_{22}^{\text{new}} = \gamma_{33}^{\text{new}} = -1,$$
$$\gamma_{11}^{\text{old}} = \gamma_{22}^{\text{old}} = \gamma_{33}^{\text{old}} = -1, \qquad \gamma_{00}^{\text{new}} = 1, \qquad \gamma_{00}^{\text{old}} = 1.$$

Therefore, the form-invariance conditions (1.20.7) for the metric (1.20.23) under the coordinate transformations of the form (1.20.3) will be fulfilled, if the functions f and Ψ obey the equations

$$\left[\Psi_t^2 - c^2\Psi_x^2 - 2c\Psi_x\Psi_t\sin\mathrm{h}\frac{wt}{c}\right]\cos\mathrm{h}^2\frac{w\Psi}{c} - \cos\mathrm{h}^2\frac{wt}{c} = 0,$$

$$f_t = -\frac{c}{w}\frac{\partial}{\partial t}\cos\mathrm{h}\frac{w\Psi}{c} + \sqrt{\Psi_t^2\cos\mathrm{h}^2\frac{w\Psi}{c} - 1}, \tag{1.20.24}$$

$$f_x = -\frac{c}{w}\frac{\partial}{\partial x}\cos\mathrm{h}\frac{w\Psi}{c} + b\sqrt{\Psi_x^2\cos\mathrm{h}^2\frac{w\Psi}{c} + \frac{1}{c^2}}.$$

To solve these equations we will use the substitution

$$\Psi = \frac{c}{w}\,\mathrm{Ar\,sinh}\,\frac{wu}{c}, \tag{1.20.25}$$

$$t = \frac{c}{w}\,\mathrm{Ar\,sinh}\,\frac{w\tau}{c} = \frac{c}{w}\,\mathrm{Ar\,cosh}\sqrt{1 + \frac{w^2\tau^2}{c^2}}.$$

Equations (1.20.24) will then be

$$u_\tau^2 - \frac{c^2 u_x^2}{1 + \dfrac{w^2\tau^2}{c^2}} - 2u_\tau u_x \frac{\omega\tau}{\sqrt{1 + \dfrac{w^2\tau^2}{c^2}}} - 1 = 0,$$

$$f_\tau = -\frac{c}{w}\frac{\partial}{\partial\tau}\sqrt{1 + \frac{w^2u^2}{c^2}} + a\sqrt{u_\tau^2 - \frac{1}{1 + \dfrac{w^2\tau^2}{c^2}}}, \tag{1.20.26}$$

$$f_x = -\frac{c}{w}\frac{\partial}{\partial x}\sqrt{1 + \frac{w^2u^2}{c^2}} + b\sqrt{u_x^2 + \frac{1}{c^2}}.$$

Hence, equations (1.20.24) are reduced by substitution (1.20.25) to the equations discussed in the previous case. This will enable us to write the desired transformations in the form

$$t_{\text{old}} = \frac{c}{w}\,\mathrm{Ar\,sinh}\,\frac{w}{c}\left[\frac{\dfrac{c}{w}\sinh\dfrac{wt_{\text{new}}}{c} + \dfrac{Vx_{\text{new}}}{c^2} + \dfrac{V}{w}\left(\cosh\dfrac{wt_{\text{new}}}{c} - 1\right)}{\sqrt{1 - V^2/c^2}}\right] + \Psi_0,$$

$$x_c = \frac{x_{\text{new}} + \dfrac{cV}{w}\sinh\dfrac{wt_{\text{new}}}{c} + \dfrac{c^2}{w}\left(\cosh\dfrac{wt_{\text{new}}}{c} - 1\right)}{\sqrt{1 - V^2/c^2}}$$

$$- \frac{c^2}{w}\left[\left\{1 + \frac{w^2}{c^2 - V^2}\left[\frac{c}{w}\sinh\frac{wt_{\text{new}}}{c} + \frac{Vx_{\text{new}}}{c^2} + \frac{V}{w}\left(\cosh\frac{wt_{\text{new}}}{c} - 1\right)\right.\right.\right.$$

$$\left.\left.\left. + \Psi_0\sqrt{1 - \frac{V^2}{c^2}}\right]^2\right\}^{1/2} - 1\right] + f_0. \tag{1.20.27}$$

The inverse transformations have the form

$$
\begin{aligned}
t_{\text{new}} &= \frac{c}{w}\,\mathrm{Ar\,sinh}\,\frac{w}{c}\left[\frac{\dfrac{c}{w}\sinh\dfrac{wt_{\text{old}}}{c} - \dfrac{Vx_{\text{old}}}{c^2} - \dfrac{V}{w}\left(\cosh\dfrac{wt_{\text{old}}}{c} - 1\right)}{\sqrt{1 - V^2/c^2}}\right. \\
&\qquad \left. - \frac{\Psi_0 - \dfrac{Vf_0}{c^2}}{\sqrt{1 - V^2/c^2}}\right],
\end{aligned}
$$

$$
\begin{aligned}
x_{\text{new}} &= \frac{x_{\text{old}} - \dfrac{cV}{w}\sinh\dfrac{wt_{\text{old}}}{c} + \dfrac{c^2}{w}\left(\cosh\dfrac{wt_{\text{old}}}{c} - 1\right)}{\sqrt{1 - V^2/c^2}} \\
&\quad - \frac{c^2}{w}\left[\left\{1 + \frac{w^2}{c^2 - V^2}\left[\frac{c}{w}\sinh\frac{wt_{\text{old}}}{c} - \frac{Vx_{\text{old}}}{c^2}\right.\right.\right. \\
&\quad - \frac{V}{w}\left(\cosh\frac{wt_{\text{old}}}{c} - 1\right) \\
&\quad \left.\left.\left. - \Psi_0\sqrt{1 - \frac{V^2}{c^2}}\right]^2\right\}^{1/2} - 1\right] - \frac{f_0 - V\Psi_0}{\sqrt{1 - V^2/c^2}}.
\end{aligned}
$$

It should be emphasized that in space-time both with metric (1.20.2) and metric (1.20.23) there can be no synchronization of clocks.

We will illustrate this by the example of the metric (1.20.23). The interval of the proper time will now be

$$\widetilde{d\tau} = dt - \frac{dx}{c}\sinh\frac{wt}{c}. \tag{1.20.28}$$

It is easily seen that this expression is no total differential. Indeed, the necessary and sufficient condition for the expression (1.20.28) to be a total differential is

$$\frac{\partial^2\tau}{\partial x\partial t} = \frac{\partial^2\tau}{\partial t\partial x}. \tag{1.20.29}$$

This guarantees that the mixed partial derivatives will be independent of the order of differentiation.

In this case, if the expressions (1.20.28) were a total differential, the partial derivatives of the first order would be

$$\frac{\partial \tau}{\partial t} = 1, \qquad \frac{\partial \tau}{\partial x} = -\frac{1}{c}\sinh\frac{wt}{c}.$$

Differentiating the first of these with respect to x, and the second with respect to t, we will have

$$\frac{\partial^2 \tau}{\partial x \partial t} = 0, \qquad \frac{\partial^2 \tau}{\partial t \partial x} = -\frac{w}{c^2}\cosh\frac{wt}{c}.$$

It follows that the condition (1.20.29) does not hold.

Since the expression (1.20.28) is no total differential, the integrals of the form

$$\tau = \int_A^B d\tilde{\tau}$$

will not be single-valued functions of points A and B, but will also be influenced by the shape of the integration path and the law of motion of the particle along this path. The integral along a closed contour will also depend on the integration path. That is why there is no way to achieve clock synchronization in this case.

We have thus discussed the relativistic uniformly accelerated reference frame, and have then posed the question of whether there exist other accelerated frames whose quadratic form is form-invariant to the quadratic form (1.20.2) of the original reference frame. Then, in all of these frames all equations of physics will be form-invariant and all physical processes in them will occur in a similar manner, so that the generalized principle of relativity will be fulfilled. As we have seen, there appeared to be infinitely many such reference frames. It is also clear that these transformations form a group: if each transformation does not change the form of the metric, then two successive transformations (or their combination) will not change it either. These transformations include the inverse and identical (unit) transformations.

That explains why we cannot formulate the generalized principle of relativity: for any noninertial reference frame we can indicate an infinite set of other noninertial frames in which the metric has the same functional form, as a result of which all the equations of physics in these frames are form-invariant. Therefore, no physical experiment can show in which of such accelerated reference frames we reside. So, if we stick to the viewpoint that unified space-time features pseudo-Euclidean geometry and consider from this viewpoint coor-

dinate systems, we could look quite differently at the principle of relativity and, most important, substantially expand the scope of special relativity.

1.21. Clock Paradox

Let us consider two inertial reference frames moving at a velocity V relative to each other. We will agree that one of the frames is at rest, and the other in motion. Suppose now that in each of them there is an identical clock. We will use them to time the interval between some two events that occur at the same point in the moving reference frame. Then, as follows from the expression (1.4.1) the clock in the frame at rest will show more time than the clock in the frame in motion. This effect has been experimentally confirmed (one of the confirmations is the growth of the lifetime of the moving μ-mesons as compared with the lifetime of the μ-mesons that are at rest relative to the observer) and at the present time this is not questioned any more. But since all inertial frames are identical, the effect is reversible, because the clock moving in the original frame will in the comoving inertial frame appear to be at rest, and the clock at rest will be in motion.

Suppose that at $t = 0$ the origins of two inertial reference frames coincide. If two observers in these reference frames checked their clocks at the origin at $t = 0$, and then parted, and met again after some time at the same point of space, what will their clocks show? The answer to this question is what is known as the solution to the so-called "clock paradox".

But the two observers in different inertial reference frames, having checked their clocks at the same point of space, will be unable later on to meet at some other point of space, since for this at least one of them would have to discontinue his inertial motion and for some time to transfer to the noninertial reference frame. And since this would violate the equality of the clocks, it is quite natural that when they met, the clock that had moved in a noninertial manner would run slower than the clock that had moved in an inertial manner all the time.

In the literature (see, e.g., [8, 11, 14, 18, 19, 64]) one can often encounter the opinion that gravitational fields and fields of force exert the same influence on the course of physical processes, with the result that a transition to a noninertial reference frame is equivalent to appearance in it of a gravitational field.

Therefore, it is allegedly possible to describe all phenomena in noninertial reference frames (including the solution to the clock paradox) applying only the general theory of relativity. This is not the case, however. Inertial forces and gravitational forces are absolutely different

in nature, since the curvature tensor for the former is identically zero and for the latter nonzero. Accordingly, the influence of the former on all physical processes can be completely removed throughout space (globally) by a passage to an inertial reference frame, whereas the influence of the latter can only be removed locally in space and not for all physical processes, but only for the simplest ones, such that their equations do not contain the curvature of space-time. That is why the description of all physical phenomena occurring in noninertial reference frames comes under the heading of special relativity and does not require a resort to the general theory of relativity.

This means, specifically, that the clock paradox can be solved within the framework of the special theory of relativity.

We will illustrate this by specific calculations. Suppose that we have two identical clocks at the same point of an inertial frame of reference. We will assume their indications to be coincident at $t = 0$. Suppose further that one clock is at all times at rest at the initial point and is thereby inertial. The other clock under a force begins at $t = 0$ to move relativistically and with uniform acceleration $a = w > 0$ and does so till $t = T_1$ according to the clock at rest. The force then stops to act on the second clock and during the time interval $T_1 < t < T_1 + T_2$ it moves uniformly. Then it gets exposed to a braking force and begins to move relativistically with uniform acceleration $a = -w$, and does so till $t = 2T_1 + T_2$, so that its velocity relative to the first clock becomes zero. The cycle reverses then, and the second clock arrives at the point where the first clock lies.

We will now work out the difference in the indications of the clocks in the inertial reference frame in which the first clock is at rest. Because the problem is symmetrical (four accelerated sections and two uniform), according to the clock at rest the clocks will meet at

$$T = 4T_1 + 2T_2. \tag{1.21.1}$$

For the second clock we similarly have

$$T' = 4T_1' + 2T_2',$$

where T_1' is the time span between the beginning and end of the acceleration as measured by the clock in motion, T_2' is the time span in proper time for the second clock during which it moves uniformly between the first and second accelerations.

Since the accelerated motion of the second clock can be represented as a continuous transition from one instantaneously comoving inertial reference frame to another, then by (4.1) we will have

$$T_1' = \int_0^{T_1} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt. \tag{1.21.2}$$

In the first phase the motion of the second clock is relativistic and uniformly accelerated without an initial velocity, and so from (1.19.12) for $0 \leqslant t \leqslant T_1$ we have

$$\sqrt{1 - \frac{v^2(t)}{c^2}} = \left[1 + \frac{w^2 t^2}{c^2}\right]^{-1/2}.$$

From (1.21.2),

$$T_1' = \frac{c}{w} \ln\left(\frac{wT_1}{c} + \sqrt{1 + \frac{w^2 T_1^2}{c^2}}\right).$$

Since for $T_1 \leqslant t \leqslant T_1 + T_2$ the second clock moves uniformly at

$$V = \frac{wT_1}{\sqrt{1 + \dfrac{w^2 T_1^2}{c^2}}},$$

we have from (1.21.2)

$$T_2' = \frac{T_2}{\sqrt{1 + \dfrac{w^2 T_1^2}{c^2}}}.$$

Correspondingly, by the meeting the second clock will indicate

$$T' = \frac{4c}{w} \ln\left(\frac{wT_1}{c} + \sqrt{1 + \frac{w^2 T_1^2}{c^2}}\right) + \frac{2T_2}{\sqrt{1 + \dfrac{w^2 T_1^2}{c^2}}}. \qquad (1.21.3)$$

Subtracting the expression (1.21.1) from (1.21.3), we will find the difference $\Delta T = T' - T$ of the indications of the clocks when they meet each other:

$$\begin{aligned} \Delta T &= T' - T \\ &= \frac{4c}{w} \ln\left[\frac{wT_1}{c} + \sqrt{1 + \frac{w^2 T_1^2}{c^2}}\right] \\ &\quad - 4T_1 + 2T_2\left[\frac{1}{\sqrt{1 + \dfrac{w^2 T_1^2}{c^2}}} - 1\right]. \end{aligned} \qquad (1.21.4)$$

It is easily seen that for any $w > 0$, $T_1 > 0$, $T_2 > 0$ the quantity ΔT will be negative. This implies that at the meeting the second clock will run slower than the first one. Consider now the same process in the reference frame in which the second clock is at rest all the time. This frame is not inertial since the second clock for some time moves non-uniformly relative the inertial reference frame associated with the first clock, and the remaining time it moves uniformly. In the first phase the second clock moves relativistically with uniform acceleration according to the law

$$x_0 = \frac{c^2}{w}\left[\sqrt{1 + \frac{w^2 t^2}{c^2}} - 1\right].$$

As follows from the expression (1.20.1), the coordinates (x, t) of the second observer in this section of the path are related to the coordinates (X, T) of the first (inertial) observer by

$$x = X - x_0 = X - \frac{c^2}{w}\left[\sqrt{1 + \frac{w^2 T^2}{c^2}} - 1\right]. \tag{1.21.5}$$

Therefore, in this section the metric of the noninertial reference frame associated with the second clock will have the form

$$ds^2 = \frac{c^2 dt^2}{1 + \dfrac{w^2 t^2}{c^2}} - \frac{2wt\, dx\, dt}{\sqrt{1 + \dfrac{w^2 T^2}{c^2}}} - dx^2 - dy^2 - dz^2. \tag{1.21.6}$$

In this frame the second clock is at rest all the time at point $x = 0$, and the first one moves along the geodesic

$$\frac{du^i}{ds} + \Gamma^i_{ml} u^m u^l = 0. \tag{1.21.7}$$

We will now find the law governing their motions. Using the relation $g_{nl} g^{nl} = \delta^l_i$, we will have

$$g^{00} = 1, \quad g^{01} = -\frac{wt}{c\sqrt{1 + \dfrac{w^2 t^2}{c^2}}}, \quad g^{11} = -\left[1 + \frac{w^2 t^2}{c^2}\right]^{-1}. \tag{1.21.8}$$

It follows from (1.21.6) and (1.21.8) that the only nonzero component of the Christoffel symbols will be

$$\Gamma^1_{00} = \frac{w}{c^2\left[1 + \dfrac{w^2 t^2}{c^2}\right]^{3/2}}.$$

Therefore, the equations of motion (1.21.7) for the first clock assume the form

$$\frac{du^0}{ds} = 0, \quad \frac{du^1}{ds} + \Gamma^1_{00} u^0 u^0 = 0. \tag{1.21.9}$$

Since $u^1 = (u^0 c) dx/dt$, using the first of (1.21.9) and also the relation $d/ds = (u^0/c)d/dt$, we will reduce the second of (1.21.9) to

$$\frac{d^2x}{dt^2} + c^2 \Gamma^1_{00} = 0.$$

Substituting the explicit expression for Γ^1_{00} gives

$$\frac{d^2x}{dt^2} + \frac{w}{\left[1 + \dfrac{w^2t^2}{c^2}\right]^{3/2}} = 0.$$

Solving this ordinary differential equation with the initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, we will obtain the law of motion of the first clock

$$x = \frac{c^2}{w}\left[1 - \sqrt{1 + \frac{w^2t^2}{c^2}}\right]. \tag{1.21.10}$$

We have thus all we need to determine the indications of both clocks by the end of the first stage of their motion. The proper time $d\tau$ of the first clock is at this stage related to the coordinate time t by

$$d\tau = \frac{1}{c}\, ds = dt \left[g_{00} + \frac{2}{c} g_{01} \frac{dx}{dt} - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2 \right]^{1/2}.$$

Substituting into this relation the expressions (1.21.6) and (1.21.10) gives

$$d\tau = dt.$$

In the first stage the proper time of the first clock thus coincides with the coordinate time, therefore, using the second formula of (1.21.5), we find that by the end of the stage the indication of the first clock τ_1 will be $\tau_1 = T_1$. Since the second clock is at rest relative the noninertial reference frame, their proper time can be found from the expression

$$\tau' = \sqrt{g_{00}}\, dt.$$

Since the first stage lasts through the interval $0 < t \leqslant T_1$ in the coor-

dinate time, by the end of it the second clock will show

$$\tau_1' = \int_0^{T_1} \sqrt{g_{00}}\, dt = \int_0^{T_1} \frac{dt}{\sqrt{1 + \dfrac{w^2 t^2}{c^2}}}$$

$$= \frac{c}{w} \ln \left[\frac{wT_1}{c} + \sqrt{1 + \frac{w^2 T_1^2}{c^2}} \right]. \qquad (1.21.11)$$

At the end of the first stage, once the velocity

$$v = \frac{wT_1}{\sqrt{1 + \dfrac{w^2 T_1^2}{c^2}}},$$

has been reached, the action of the accelerating force of the second clock discontinues, and it moves one uniformly. Therefore, at the second stage the law of motion of the origin of the system associated with the clock will be

$$x = vt.$$

Since at this stage the motion of both clocks is uniform and rectilinear, both associated reference frames will be inertial. Therefore, it might appear that the coordinates and time in these frames should be related by the Lorentz transformation (1.3.9). But here in the second stage the metric of both reference frames will be Galilean

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2, \qquad (1.21.12)$$

and so at $t = T_1$ the metric tensor in the reference frame associated with the second clock must abruptly change from the value

$$g_{00} = \frac{1}{1 + \dfrac{w^2 T_1^2}{c^2}} < 1, \quad g_{01} = -\frac{wT_1}{\sqrt{1 + \dfrac{w^2 T_1^2}{c^2}}},$$

$$g_{11} = -1, \quad g_{22} = -1, \quad g_{13} = -1$$

to the value

$$g_{00} = 1, \quad g_{01} = 0, \quad g_{11} = g_{22} = g_{33} = -1.$$

Since the metric tensor of space-time must be a continuous quantity, we must match the coordinates in the first and second stages. There are two ways of doing this. The first one is traditional and consists in

changing by some law $t = F(x, t')$ the origin and the rate of time flow of the coordinate clock in a noninertial reference frame associated with the second observer so that the metric of the frame should become diagonal and its space coordinate should be time-independent. Such reference frames are known in the literature as rigid ones. There can then be a rigid reference frame with a certain set of integration constants, such that the component g_{00} when the frame is in noninertial motion would take on the value $g_{00} = 1$ at least at two predetermined moments of time. If we take the first of them to be the origin of noninertial motion, and the second its end, then the metric of the rigid frame can go into the Galilean metric (1.21.12) in a continuous manner in inertial motion sections. Such a path is rather complicated, however. Moreover, there is no telling a priori that an appropriate transformation $t = F(x, t')$ can be found for any noninertial reference frame.

Therefore, for the metric tensor to be continuous when the reference frame passes from noninertial motion to inertial one we will make use of the new possibilities that have opened up when we have expanded the class of inertial reference systems up to that of generalized inertial frames. The generalized inertial reference frames, as it has been shown in some detail in Sections 12-16, are absolutely equal as far as description of physical phenomena with Lorentzian inertial reference frames is concerned, and they can have nondiagonal metrics. We will therefore require that at $t = T_1$ the metric of a noninertial frame associated with the second clock would pass continuously into the metric of a generalized inertial reference frame. For this purpose, it is sufficient in the second stage to relate the coordinates and time of the second observer (X, T) to the coordinates (x, t) by

$$x = X - x_0(T) = X - VT, \quad t = T,$$

where

$$V = \frac{wT_1}{\sqrt{1 + \dfrac{w^2 T_1^2}{c^2}}}.$$

In this case, in the interval of coordinate time $T_1 < t < T_1 + T_2$ the reference frame of the second clock has the metric

$$ds^2 = c^2\left(1 - \frac{V^2}{c^2}\right)dt^2 - 2Vdxdt - dx^2 - dy^2 - dz^2. \qquad (1.21.13)$$

It becomes obvious that at $t = T_1$ the metric (1.21.6) of a noninertial frame goes into the metric of a generalized inertial reference frame

(1.21.13) in a continuous manner. Using the equation of geodesics (1.21.7), we will find the law of motion of the first clock relative to the second one. Since in the reference frame with the metric (1.21.13) all the components $\Gamma^i_{nl} = 0$, we will find, using the initial condition $-\dot{x} = V$ at $t = T$, that

$$\frac{dx}{dt} = -V.$$

Therefore, in the first stage too the proper time $d\tau$ of the first clock

$$d\tau = \frac{1}{c}\,ds = dt\left[g_{00} + \frac{2}{c}\,g_{01}\frac{dx}{dt} - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right]^{1/2}$$

coincides with the coordinate time: $d\tau = dt$.

Since in this phase of motion the coordinate time t varies in the interval $T_1 < t < T_1 + T_2$, we will have by the first clock $\tau_2 = T_2$. On the other hand, the second clock is at rest ($dx/dt = 0$) relative to the reference frame we have chosen, and hence their proper time is

$$d\tau' = \sqrt{g_{00}}\,dt.$$

Hence

$$\tau_2' = \int_{T_1}^{T_1+T_2} \sqrt{g_{00}}\,dt = T_2\sqrt{1 - V^2/c^2} = \frac{T_2}{\sqrt{1 + \dfrac{w^2 T_1^2}{c^2}}}.$$

By the symmetry of the problem, these data are sufficient to determine what the clocks will show when they meet after the motion cycle is over. Indeed, the indication of the first clock, τ, in the frame of the second clock can be found from the relation

$$\tau = 4\tau_1 + 2\tau_2.$$

Likewise, the indication of the second clock τ' in the same reference frame will be

$$\tau' = 4\tau_1' + 2\tau_2'.$$

Substituting into these relations the expressions for τ_1, τ_2, τ_1' and τ_2', we get

$$\tau' = \frac{4c}{w}\ln\left[\frac{wT_1}{c} + \sqrt{1 + \frac{w^2T_1^2}{c^2}}\right] + \frac{2T_2}{\sqrt{1 + \dfrac{w^2T_1^2}{c^2}}},$$

$$\tau = 4T_1 + 2T_2. \qquad (1.21.14)$$

It follows that the difference in the indications of the clocks when they meet, i.e., $\Delta\tau = \tau' - \tau$, in the frame of the second clock, will be

$$\Delta\tau = \frac{4c}{w} \ln\left[\frac{wT_1}{c} + \sqrt{1 + \frac{w^2 T_1^2}{c^2}}\right]$$

$$- 4T_1 + 2T_2\left[\frac{1}{\sqrt{1 + \dfrac{w^2 T_1^2}{c^2}}} - 1\right]. \qquad (1.21.15)$$

Comparing the expressions (1.21.14) and (1.21.15) with the expressions (1.21.1), (1.21.3) and (1.21.4), it is evident that calculations in the reference frame associated with the first clock yield the same result as in the frame of the second clock: the second clock, after it completes the entire cycle and meets the first clock, will appear to run slower.

The value of the difference in the indications of the clocks depends on w, T_1 and T_2. In the special case where the acceleration w tends to infinity, and the time of the accelerated motion $T_1 \to 0$, so that the flow rate of the second clock

$$V = wT_1\left[1 + \frac{w^2 T_1^2}{c^2}\right]^{-1/2}$$

after the acceleration would be a finite quantity, from the expression (1.21.15) we have

$$\Delta\tau = 2T_2[\sqrt{1 - \frac{V^2}{c^2}} - 1].$$

This relation shows that the clock that undergoes an instantaneous acceleration, and then moves inertially indefinitely is not identical with the inertial clock. We have thus shown that both in the reference frame of the first clock and in the frame of the second clock the calculation of the difference in their indications gives the same result. This implies that the slowing of the clock that has moved noninertially relative to the clock that has moved inertially is an absolute effect, which is independent of the choice of the reference frame.

1.22. Relation Between Coordinate and Physical Quantities

As it has already been mentioned repeatedly throughout the book, all physical phenomena can be described equivalently, physically and mathematically, in arbitrary permissible coordinates. When we describe physical phenomena, in arbitrary coordinates, two kinds of quantities emerge, coordinate and physical ones. We have already encountered them when we discussed the coordinate and physical velocities in an arbitrary inertial reference frame.

To understand the difference between these two kinds of quantities and to learn how to correlate physical and coordinate quantities in an arbitrary permissible reference frame, we consider a simple case: we will see how in an arbitrary inertial reference frame the coordinate time and length are related to the physical time and length. Since these two concepts – time and length – play a fundamental role in physics (all physical processes occur in space and time), the law that puts into correspondence to the coordinate quantities of the physical ones will not be just a special case, it will rather be a general law, one that holds for all physical quantities.

In an arbitrary curvilinear system of coordinates of pseudo-Euclidean space-time we will consider the interval

$$ds^2 = g_{ik}\,dx^i dx^k. \tag{1.22.1}$$

This quantity is an invariant under all permissible transformations of the coordinates of space-time. Isolating in (1.22.1) the terms with zero indices, we will obtain

$$\begin{aligned} ds^2 &= c^2 g_{00}dt^2 + 2g_{0\alpha}c\,dt\,dx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta \\ &= c^2\left[\sqrt{g_{00}}\,dt + \frac{g_{0\alpha}dx^\alpha}{c\sqrt{g_{00}}}\right]^2 - \kappa_{\alpha\beta}dx^\alpha dx^\beta, \end{aligned} \tag{1.22.2}$$

where $\kappa_{\alpha\beta} = -g_{\alpha\beta} + g_{0\alpha}g_{0\beta}/g_{00}$ is the three-dimensional metric tensor.

Since the expression for ds^2 is invariant under all permissible coordinate transformations, the division of ds^2 into two such parts will be invariant as well: under all permissible coordinate transformations the quantity

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0\alpha}dx^\alpha}{c\sqrt{g_{00}}} \tag{1.22.3}$$

has a time-like character, and the quantity

$$dR^2 = \kappa_{\alpha\beta}dx^\alpha dx^\beta \tag{1.22.4}$$

has a space-time character. These quantities are physical, therefore the quantity (1.22.3) can quite naturally be referred to as physical time, and the quantity (1.22.4) as the square of the physical length. They are directly connected with the interval (1.22.1) by a relation similar to the expression (1.2.13) for the interval in Galilean inertial reference frames

$$ds^2 = c^2 d\tau^2 - dR^2. \tag{1.22.5}$$

It is worth noting that in the general case the expression (1.22.3) is not a total differential. The condition for the expression (1.22.3) to become a total differential, as it has already been mentioned above, is closely linked with the possibility of clock synchronization in a given reference frame: clocks can only be synchronized in reference frames in which $d\tau$ is a total differential.

Reference frames in which the interval has the form

$$ds^2 = c^2 dT^2 - dX^2 - dY^2 - dZ^2,$$

give the simplest connection between physically measurable quantities $d\tau$ and dR and the coordinate quantities dT, dX, dY, and dZ:

$$d\tau = dT, \quad dR^2 = dX^2 + dY^2 + dZ^2.$$

In other reference frames the connection is more complicated.

We have thus established that the physically measurable time $d\tau$ is a linear combination of the coordinate time dt and the coordinate differential dx^α, the coefficients of this combination are built up of the components of the metric tensor. The square of a physically measurable length dR^2 is a quadratic form of the coordinate differentials. Then, taking $cd\tau$ to be the zero component of the measurable differential dX^i and dR^2 a quadratic form composed of its space components, we have

$$dX^{\bar{i}} = \lambda^{\bar{i}}_l\, dx^l,$$

where $\lambda^{\bar{i}}_l$ are some basis vectors dependent on the metric. In the literature the sixteen components $\lambda^{\bar{i}}_l$ are called the tetrad.

The interval in physically measurable differentials is a diagonal quantity, and so to define the components of the basis vectors $\lambda^{\bar{i}}_l$ we have the following equation:

$$ds^2 = \gamma_{\bar{i}\bar{k}}\, dX^{\bar{i}} dX^{\bar{k}} = \lambda^{\bar{i}}_l \lambda^{\bar{k}}_m \gamma_{\bar{i}\bar{k}}\, dx^l dx^m = g_{lm}\, dx^l dx^m.$$

Hence the metric tensor g_{lm} is related to the tetrad by

$$g_{lm} = \lambda^{\bar{i}}_l \lambda^{\bar{k}}_m \gamma_{\bar{i}\bar{k}}. \tag{1.22.6}$$

In consequence, to determine the sixteen components of the basis vectors $\lambda^{\bar{i}}_{l}$ we will have only ten equations. How are we to derive them? Recall that we have already established the relation between the physical time $d\tau$ and the coordinate quantities dx^{l}:

$$d\tau = \sqrt{g_{00}}dt + \frac{g_{0\alpha}dx^{\alpha}}{c\sqrt{g_{00}}} = \lambda^{\bar{0}}_{l}dx^{l}.$$

From this we can now determine the four components $\lambda^{\bar{0}}_{l}$

$$\lambda^{\bar{0}}_{l} = \frac{g_{0l}}{\sqrt{g_{00}}}, \quad l = 0, 1, 2, 3. \tag{1.22.7}$$

Substituting this expression into (1.22.6) gives

$$\begin{aligned} (\lambda^{\bar{1}}_{0})^{2} + (\lambda^{\bar{2}}_{0})^{2} + (\lambda^{\bar{3}}_{0})^{2} &= 0, \\ \lambda^{\bar{1}}_{0}\lambda^{\bar{1}}_{\alpha} + \lambda^{\bar{2}}_{0}\lambda^{\bar{2}}_{\alpha} + \lambda^{\bar{3}}_{0}\lambda^{\bar{3}}_{\alpha} &= 0, \\ -\kappa_{\alpha\beta} &= \lambda^{\bar{\mu}}_{\alpha}\lambda^{\bar{\nu}}_{\beta}\gamma_{\bar{\mu}\bar{\nu}}. \end{aligned} \tag{1.22.8}$$

From the first of these we have

$$\lambda^{\bar{1}}_{0} = \lambda^{\bar{2}}_{0} = \lambda^{\bar{3}}_{0} = 0.$$

The second equation of (1.22.8) will then be satisfied identically.

Therefore, to determine the remaining nine components $\lambda^{\bar{\nu}}_{\beta}$ we have six equations

$$-\kappa_{\alpha\beta} = \lambda^{\bar{\mu}}_{\alpha}\lambda^{\bar{\nu}}_{\beta}\gamma_{\mu\bar{\nu}}. \tag{1.22.9}$$

These equations, however, are equations of the second degree in the unknowns $\lambda^{\bar{\nu}}_{\beta}$, therefore the count of the number of equations and of the number of unknowns gives us yet no clue as to the number of their solutions. To determine the components $\lambda^{\bar{\nu}}_{\beta}$ we will proceed as follows. We consider the segment at rest aligned along some axis, say, the x-axis. Since then $dy = dz = 0$, the square of its length dR^{2} can be written as

$$(dX^{\bar{1}})^{2} = dR^{2} = \kappa_{\alpha\beta}dx^{\alpha}dx^{\beta}.$$

Considering that $dX^{\bar{1}} = \lambda^{\bar{1}}_{1}dx$ at $dy = dz = 0$, we get

$$\lambda^{\bar{1}}_{1} = \sqrt{\kappa_{11}}. \tag{1.22.10}$$

Aligning in succession the segment being measured along the other

axes, we will find

$$\lambda_{2}^{\bar{2}} = \sqrt{\kappa_{22}}, \quad \lambda_{3}^{\bar{3}} = \sqrt{\kappa_{33}}. \tag{1.22.11}$$

The remaining six components $\lambda_{\beta}^{\bar{\alpha}}$ will then obey the six equations

$$\begin{aligned}
&(\lambda_1^{\bar{2}})^2 + (\lambda_1^{\bar{3}})^2 = 0, \quad (\lambda_2^{\bar{1}})^2 + (\lambda_2^{\bar{3}})^2 = 0, \quad (\lambda_3^{\bar{1}})^2 + (\lambda_3^{\bar{2}})^2 = 0,\\
&\lambda_1^{\bar{1}}\lambda_2^{\bar{1}} + \lambda_1^{\bar{2}}\lambda_2^{\bar{2}} + \lambda_1^{\bar{3}}\lambda_2^{\bar{3}} = 0,\\
&\lambda_1^{\bar{1}}\lambda_3^{\bar{1}} + \lambda_1^{\bar{2}}\lambda_3^{\bar{2}} + \lambda_1^{\bar{3}}\lambda_3^{\bar{3}} = 0,\\
&\lambda_2^{\bar{1}}\lambda_3^{\bar{1}} + \lambda_2^{\bar{2}}\lambda_3^{\bar{2}} + \lambda_2^{\bar{3}}\lambda_3^{\bar{3}} = 0.
\end{aligned} \tag{1.22.12}$$

It follows from the first three equations of the system that $\lambda_1^{\bar{2}} = \lambda_1^{\bar{3}} = \lambda_2^{\bar{1}} = \lambda_2^{\bar{3}} = \lambda_3^{\bar{1}} = \lambda_3^{\bar{2}} = 0$. It is easily seen that the remaining three equations of the system (1.22.12) also hold identically. We have thus found all the components $\lambda_{\beta}^{\bar{\alpha}}$.

As we have established, the physical time and length are a local four-vector; they are related to the coordinate differentials dt and dx^{α} by

$$dX^{\bar{i}} = \lambda_{l}^{\bar{i}}\, dx^{l}.$$

Space and time are fundamental concepts; it follows therefore that this relation has a general nature. It is a law by which the coordinate four-vector a^{i} is put into correspondence to the physical four-vector

$$A^{\bar{i}} = \lambda_{l}^{\bar{i}}\, a^{l}. \tag{1.22.13}$$

Generalization of the law to the case of the tensor of arbitrary valence is a straightforward exercise. Let $q^{lm\ldots n}$ be a coordinate tensor, then the physically measurable tensor $Q^{\overline{ip}\ldots\bar{k}}$ can be obtained by the law

$$Q^{\bar{i}\,\bar{p}\ldots\bar{k}} = \lambda_{l}^{\bar{i}}\lambda_{m}^{\bar{p}}\ldots\lambda_{n}^{\bar{k}}\, q^{lm\ldots n}. \tag{1.22.14}$$

Covariant physically measurable quantities will be

$$A_{\bar{i}} = \gamma_{\bar{i}\,\bar{k}} A^{\bar{k}},$$

i.e., to raise and lower their indices we can use the Galilean metric tensor $\gamma_{\bar{i}\,\bar{k}}$.

Since to the invariant four-volume

$$d\Omega = \sqrt{-g}\, c\; dt\, dx\, dy\, dz$$

we put into correspondence the physical four-volume in Galilean coor-

dinates

$$d\Omega = c\, dT\, dX\, dY\, dZ,$$

whence it follows that the determinant of the metric tensor $\gamma_{\bar{i}\bar{k}}$ is unity.

1.23. Equations and Relations of Mechanics in Arbitrary Inertial Reference Frame

We now write the equations of mechanics in arbitrary inertial reference frame whose metric has the form (12.11). Since in this case we have

$$ds = c\left[g_{00}\left(1 - \frac{v^x}{c_1}\right)\left(1 - \frac{v^x}{c_2}\right) - \frac{(v^y)^2 + (v^z)^2}{c^2}\right]^{1/2} dt,$$

$$u^0 = \frac{c\, dt}{ds}, \quad u^\alpha = \frac{v^\alpha}{c} u^0, \quad F^\alpha = f^\alpha \frac{u^0}{c}, \tag{1.23.1}$$

we will obtain from the generally covariant equations (1.18.2)

$$m_0 \frac{d}{dt} \frac{v^\alpha}{\sqrt{g_{00}\left(1 - \dfrac{v^x}{c_1}\right)\left(1 - \dfrac{v^x}{c_2}\right) - \dfrac{1}{c^2}(v^y)^2 - \dfrac{1}{c^2}(v^z)^2}} = f^\alpha. \tag{1.23.2}$$

Using the definition of the four-momentum of a particle $p^i = m_0 c u^i$ we write these equations in the form

$$\frac{d}{dt} p^\alpha = f^\alpha. \tag{1.23.3}$$

Solving the equations (1.23.2) or (1.23.3), we can determine all the coordinate quantities of interest as functions of any other coordinate quantities $x(t)$, $y(t)$, $z(t)$, $v^\alpha(t)$, $p^i(t)$, $p^i(x, y, z)$, and so on. We can also find the relation between the coordinate time and the physical time (the proper time of an observer), and then express all the resultant physical quantities as functions of physical time. Thus, description of mechanical phenomena in an arbitrary inertial reference frame enables us to obtain all the information of interest concerning mechanical motion.

We now show that the formalism presented above, which allows us to put into correspondence to coordinate quantities some physical quantities, leads to relations that are in perfect agreement with experiment.

We know from experiment that the energy and momentum of a particle are related to its physical velocity V by

$$E = \frac{m_0 c^2}{\sqrt{1 - V^2/c^2}}, \quad P^{\bar{\alpha}} = \frac{m_0 V^{\bar{\alpha}}}{\sqrt{1 - V^2/c^2}}, \tag{1.23.4}$$

and these quantities constitute the four-vector $P^{\bar{i}} = (E/c, P^{\bar{\alpha}})$. Solving (1.23.3), we will obtain the coordinate four-vector of the particle's momentum, when the particle moves with the coordinate velocity v^{α}:

$$P^i = m_0 c u^i, \tag{1.23.5}$$

where

$$u^i = \left(\frac{cdt}{ds}, \; v^{\alpha} \frac{dt}{ds} \right).$$

Using the definitions (1.22.5), (1.11.7), and (1.4.13), we will find

$$\frac{cd\tau}{ds} = \frac{1}{\sqrt{1 - (dR^2)/c^2 d\tau^2}} = \frac{1}{\sqrt{1 - V^2/c^2}}. \tag{1.23.6}$$

From the coordinate four-vector u^i of velocity we have

$$u^0 = \frac{1}{\sqrt{1 - V^2/c^2}} \frac{dt}{d\tau}, \quad u^{\alpha} = \frac{1}{c\sqrt{1 - V^2/c^2}} \frac{dx^{\alpha}}{d\tau}. \tag{1.23.7}$$

From (1.23.14) and (1.23.5) we can then write

$$E = cP^{\bar{0}} = m_0 c^2 u^i \lambda^{\bar{0}}_i, \quad P^{\bar{\alpha}} = m_0 c u^i \lambda^{\bar{\alpha}}_i. \tag{1.23.8}$$

Since in this case the nonzero components of the tetrad $\lambda^{\bar{i}}_k$ have the form

$$\lambda^{\bar{0}}_i = \frac{g_{0i}}{\sqrt{g_{00}}}, \qquad \lambda^{\bar{1}}_1 = \sqrt{\kappa_{11}}, \quad \lambda^{\bar{2}}_2 = \lambda^{\bar{3}}_3 = 1, \tag{1.23.9}$$

the relations (1.23.7) and (1.23.8) yield the following expression for

the energy:

$$E = m_0 c^2 u^i \lambda_i^{\bar{0}} = \frac{m_0 c^2}{\sqrt{1 - V^2/c^2}} \left[\frac{dt}{d\tau} \lambda_0^{\bar{0}} + \frac{dx^\alpha}{c\,d\tau} \lambda_\alpha^{\bar{0}} \right]$$

$$= \frac{m_0 c^2}{\sqrt{1 - V^2/c^2}} \; \frac{\left[\sqrt{g_{00}}\, dt + \dfrac{g_{0\alpha} dx^\alpha}{c\sqrt{g_{00}}} \right]}{d\tau} = \frac{m_0 c^2}{\sqrt{1 - V^2/c^2}} .$$

We have thus derived for the energy the well-known expression, which agrees with experimental results. In much the same way, we can obtain expressions for the components $P^{\bar{\alpha}}$ of the momentum

$$P^{\bar{1}} = m_0 c u^i \lambda_i^{\bar{1}} = m_0 c u^1 \lambda_1^{\bar{1}} = \frac{m_0}{\sqrt{1 - V^2/c^2}} \; \frac{\sqrt{\kappa_{11}}\, dx}{d\tau}$$

$$= \frac{m_0 V^2 .}{\sqrt{1 - V^2/c^2}} ,$$

$$P^{\bar{2}} = m_0 c u^i \lambda_i^{\bar{2}} = m_0 c u^2 \lambda_2^{\bar{2}}$$

$$= \frac{m_0}{\sqrt{1 - V^2/c^2}} \; \frac{dY}{d\tau} = \frac{m_0 V^y}{\sqrt{1 - V^2/c^2}} ,$$

$$P^{\bar{3}} = m_0 c u^i \lambda_i^{\bar{3}} = m_0 c u^3 \lambda_3^{\bar{3}}$$

$$= \frac{m_0}{\sqrt{1 - V^2/c^2}} \; \frac{dZ}{d\tau} = \frac{m_0 V^z}{\sqrt{1 - V^2/c^2}} .$$

In our case the law of construction of physical quantities thus leads to correct expressions for the energy and momentum of the particle.

As we have seen, physical quantities are determined not only by corresponding coordinate quantities but also by the metric, which is here included into the tetrad of the representation. This again stresses the close link of physical quantities with the geometry of space-time.

1.24. Equations of Electrodynamics in Arbitrary Inertial Reference Frame

When handling a wide variety of problems of electrodynamics we can also use any permissible coordinate systems.

The generally covariant equations of the electromagnetic field in arbitrary coordinates are

$$\nabla_k F^{ik} = -\frac{4\pi}{c} j^i,$$

$$\nabla_l F_{ik} + \nabla_i F_{kl} + \nabla_k F_{li} = 0, \tag{1.24.1}$$

where F_{ik} is the tensor of the electromagnetic field, j^i is the four-vector of the current. Because the tensor of the electromagnetic field has the property of antisymmetry, its relation with the four-vector of the potential A_i retains the normal form

$$F_{ik} = \nabla_i A_k - \nabla_k A_i = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}. \tag{1.24.2}$$

By virtue of the tensor F_{ik} being antisymmetric, the equations of the electromagnetic field (1.24.1) can be written in another form

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} [\sqrt{-g}\, F^{ik}] = \frac{4\pi}{c} j^i, \quad \frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0. \tag{1.24.3}$$

As we see with allowance for (1.24.2) the last equation of the system is valid identically; therefore, in what follows we will often omit it.

Before constructing the generally covariant four-vector of the current, let us recall some properties of the Dirac delta-function (for more detail see [20-21]).

When generalizing the concepts of mass density and charge density as applied to particles, physicists found it necessary to introduce into treatments generalized functions. And since these functions behaved unlike conventional functions, mathematicians were suspicious about them. But the necessity and usefulness of the generalizations of the concept of the function was proved with time, and so mathematicians soon constructed a theory of generalized functions. Among generalized functions the most widely used in physics is Dirac's delta-function.

The unidimensional Dirac's delta-function is defined by the requirements

$$\delta(x) = \begin{cases} \infty & \text{at} \quad x = 0, \\ 0 & \text{at} \quad x \neq 0, \end{cases}$$

$$\int_a^b \delta(x - x_0)\, dx = \begin{cases} 1, & x_0 \in (a, b), \\ 1/2, & x_0 = a \text{ or } x_0 = b, \\ 0, & x_0 \notin [a, b]. \end{cases}$$

From these definitions follow the properties of Dirac's delta-function

$$\int_b^c f(x)\, \delta(x - a)\, dx = f(a), \quad a \in (b, c),$$

$$\delta(-x) = \delta(x), \quad \delta(ax) = \frac{\delta(x)}{|a|}, \quad \delta(f(x)) = \sum_n \frac{\delta(x - x_n)}{\left|\dfrac{df}{dx}\right|}, \tag{1.24.4}$$

where $f(x_n) = 0$.

Generalization of Dirac's delta-function to three dimensions gives

$$\delta(x) = \delta(x^1)\, \delta(x^2)\, \delta(x^3).$$

Therefore, in any curvilinear system of coordinates we have

$$\int \delta(x)\, dx^1 dx^2 dx^3 = 1. \tag{1.24.5}$$

It follows that the three-dimensional delta-function possesses the properties of the scalar of density weight 1 under three-dimensional coordinate transformations. To prove this it is sufficient to change the variables in the integral (1.24.5) and use the last of the properties (1.24.4) of the delta-function.

In arbitrary coordinates, the four-vector of the current, as we have seen, has the form

$$j^i = \rho_0 c u^i. \tag{1.24.6}$$

This four-vector obeys the continuity equation

$$\Delta_i j^i = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} [\sqrt{-g}\, j^i] = 0. \tag{1.24.7}$$

This equation can be written as

$$\frac{c}{\sqrt{-g}}\left[\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x^\alpha}(\rho\, v^\alpha)\right] = 0. \tag{1.24.8}$$

This relation is a generally covariant one. By this equation, the quantity

$$q = \int \rho\, dV = \text{const}$$

is time-independent. In arbitrary coordinates we thus have

$$j^i = \left(\frac{\rho c}{\sqrt{-g}}, \frac{\rho v^\alpha}{\sqrt{-g}}\right). \tag{1.24.9}$$

In a three-dimensional space with the metric $\kappa_{\alpha\beta}$ it is more convenient to use the charge density ρ^* related to the conserved density ρ by

$$\rho^* = \frac{\rho}{\sqrt{\kappa}}. \tag{1.24.10}$$

In this case the relation between the charge de at the element of the three-dimensional volume $\sqrt{\kappa}\, dV$ and the density ρ^* assumes the three-dimensional invariant form

$$de = \rho^* \sqrt{\kappa}\, dx^1\, dx^2\, dx^3. \tag{1.24.11}$$

For the point charge we then have

$$\rho^* = \frac{q}{\sqrt{\kappa}}\delta(x^1)\delta(x^2)\delta(x^3). \tag{1.24.12}$$

And so, on the physical side, the density ρ^* is the charge per unit invariant volume in a three-dimensional space with the metric $\kappa_{\alpha\beta}$. By definition (1.24.10) the four-vector of the current (1.24.6) can be written in terms of ρ^*

$$j^i = \left[\frac{c\rho^*}{\sqrt{g_{00}}}, \frac{\rho^* v^\alpha}{\sqrt{g_{00}}}\right]. \tag{1.24.13}$$

Since we are rather free in our choice of coordinates, we can select the most convenient for the solution of any particular problem. In solving these equations, we can define all the coordinate quantities of interest to us as functions of the coordinate quantities x, y, z, t. Using the relations of Section 22, we can then find the dependence of physical quantities on the physical coordinates X, Y, Z and time T. Cor-

respondingly, description of electromagnetic events in arbitrary coordinates enables us to obtain all the information about a phenomenon that is of interest to us.

We will now show for an arbitrary inertial reference frame that such a description yields a relation between the physical quantities in full conformity with experiment.

We will first find the expression for the components of the physical four-vector of the current. As we know already, the physical four-vector of the current, just like any other four-vector, is related to the coordinate four-vector by

$$j^{\bar{i}} = \lambda^{\bar{i}}_n j^n. \tag{1.24.14}$$

Using the expressions (1.23.9) for the components of the tetrad $\lambda^{\bar{i}}_n$, we will get

$$j^{\bar{0}} = \rho^0 cu^i \lambda^{\bar{0}}_i = \rho_0 c \frac{dx^i}{ds} \lambda^{\bar{0}}_i = \rho_0 c \frac{d\tau}{ds} = \frac{\rho_0 c}{\sqrt{1 - V^2/c^2}}$$
$$= \rho c,$$

$$j^{\bar{\alpha}} = \rho_0 cu^i \lambda^{\bar{\alpha}}_i = \rho_0 \frac{dX^{\bar{\alpha}}}{d\tau} = \frac{\rho_0 V^\alpha}{\sqrt{1 - V^2/c^2}} = \rho V^\alpha. \tag{1.24.15}$$

The components of the physical four-vector of the current thus obey the well-known relation

$$j^{\bar{\alpha}} = j^{\bar{0}} \frac{V}{c}.$$

We now find the law of coordinate transformation for the four-vector of the current (1.24.9) in transition from one inertial reference frame (1.13.8) to another. Under coordinate transformation, the four-vector is known to be transformed by the tensor law

$$j^i_{\text{new}} = \frac{\partial x^i_{\text{new}}}{\partial x^l_{\text{old}}} j^l_{\text{old}} (x_{\text{old}}(x_{\text{new}})).$$

We get

$$j^0_{\text{new}} = \frac{\partial t_{\text{new}}}{\partial t_c} j^0_{\text{old}} + \frac{\partial t_{\text{new}}}{\partial x_c} cj^1_{\text{old}},$$

$$j^1_{\text{new}} = \frac{1}{c} \frac{\partial x_{\text{new}}}{\partial t_{\text{old}}} j^0_{\text{old}} + \frac{\partial x_{\text{new}}}{\partial x_{\text{old}}} j^1_{\text{old}}.$$

Using the formulas for the transformation of coordinates and time (1.13.8), we obtain

$$j^0_{\text{new}} = \frac{j^0_{\text{old}} + \dfrac{vc}{c_1 c_2} j^1_{\text{old}}}{\sqrt{\left(1+\dfrac{v}{c_1}\right)\left(1+\dfrac{v}{c_2}\right)}}, \quad j^1_{\text{new}} = \frac{\left(1+\dfrac{v}{c_1}+\dfrac{v}{c_2}\right) j^1_{\text{old}} - \dfrac{v}{c} j^0_{\text{old}}}{\sqrt{\left(1+\dfrac{v}{c_1}\right)\left(1+\dfrac{v}{c_2}\right)}}. \tag{1.24.16}$$

We now find the law of transformation of the physical components of the four-vector of the current. Using the expressions (1.24.14) and (1.24.16), and also the relation (1.15.4) – (1.15.7) between the coordinate and physical velocities of one reference frame relative to another one, we will arrive at the transformation formulas

$$\bar{j}^{\bar{0}}_{\text{new}} = \frac{\bar{j}^{\bar{0}}_{\text{old}} - \dfrac{V}{c} \bar{j}^{\bar{1}}_{\text{old}}}{\sqrt{1 - V^2/c^2}}, \quad \bar{j}^{\bar{1}}_{\text{new}} = \frac{\bar{j}^{\bar{1}}_{\text{old}} - \dfrac{V}{c} \bar{j}^{\bar{0}}_{\text{old}}}{\sqrt{1 - V^2/c^2}},$$

which are well known from experiment.

Let us now reduce Maxwell's equations (1.24.3) to "vector" form. For this purpose, in a three-dimensional space with the metric $\kappa_{\alpha\beta}$ we will introduce the unit skew-symmetric tensors $E^{\alpha\beta\gamma}$ and $E_{\alpha\beta\gamma}$:

$$E^{\alpha\beta\gamma} = \frac{1}{\sqrt{\kappa}} e^{\alpha\beta\gamma}, \quad E_{\alpha\beta\gamma} = \sqrt{\kappa}\, e_{\alpha\beta\gamma},$$

where κ is the determinant of the metric tensor $\kappa_{\alpha\beta}$,

$$e_{\alpha\beta\gamma} = \begin{cases} 0 & \text{when there are coincident indices,} \\ \pm 1 & \text{if all indices are different,} \end{cases}$$

Here $e_{123} = -e_{213} = 1$.

We will also introduce the notation for differential operations that generalize the operations curl **A** and div **A** to the case of a three-dimensional Riemannian space with the metric $\kappa_{\alpha\beta}$:

$$(\text{curl } \mathbf{A})^\alpha = E^{\alpha\beta\gamma} \frac{\partial}{\partial x^\beta} A_\gamma, \quad \text{div } \mathbf{A} = \frac{1}{\sqrt{\kappa}} \frac{\partial}{\partial x^\alpha} [\sqrt{\kappa} A^\alpha]. \tag{1.24.17}$$

Using these operations we can write Maxwell's equations (1.24.3) in "vector" form. We introduce the coordinate "vectors" of the magnetic induction **B** and of the electric displacement **D**

$$B^{\alpha} = -\frac{1}{2} E^{\alpha\beta\gamma} F_{\beta\gamma}, \quad D^{\alpha} = -\sqrt{g_{00}}\, F^{0\alpha}, \tag{1.24.18}$$

and also the coordinate "vectors" of the electric and magnetic fields

$$E_{\alpha} = F_{0\alpha}, \quad H_{\alpha} = -\frac{\sqrt{g_{00}}}{2} E_{\alpha\beta\gamma} F^{\beta\gamma}. \tag{1.24.19}$$

The "vectors" (1.24.18) and (1.24.19) are not independent:

$$\mathbf{D} = \frac{\mathbf{E}}{\sqrt{g_{00}}} + [\mathbf{H}\,\mathbf{n}], \quad \mathbf{B} = \frac{\mathbf{H}}{\sqrt{g_{00}}} + [\mathbf{n}\,\mathbf{E}], \tag{1.24.20}$$

where

$$n_{\alpha} = -\frac{g_{0\alpha}}{g_{00}}.$$

Separating the space and time tensor indices in the second group of Maxwell's equations (1.24.3), we will get

$$\partial_0 F_{\gamma\beta} + \partial_\gamma F_{\beta 0} + \partial_\beta F_{0\gamma} = 0.$$

Hence

$$\partial_0 F_{\gamma\beta} + \partial_\beta E_\gamma - \partial_\gamma E_\beta = 0.$$

We now multiply this by $\sqrt{\kappa}\, E^{\alpha\beta\gamma}$ and take into account that

$$\partial_n (\sqrt{\kappa}\, E^{\alpha\beta\gamma}) = \partial_n\, e^{\alpha\beta\gamma} = 0.$$

The resultant relation we can write by (1.24.17) in the form

$$-\frac{1}{c\sqrt{\kappa}} \frac{\partial}{\partial t} [\sqrt{\kappa}\mathbf{B}] = \operatorname{curl} \mathbf{E}. \tag{1.24.21}$$

Similarly, if we multiply by $\sqrt{\kappa}\, E^{\alpha\beta\gamma}$ the equations

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

and identically transform the result, we obtain

$$\operatorname{div} \mathbf{B} = 0.$$

From the first group of Maxwell's equations (1.24.3) at $i = 0$ we have

$$\operatorname{div} \mathbf{D} = 4\pi\rho^*. \tag{1.24.22}$$

Analogously, at $i = \alpha$ we get

$$\frac{1}{c\sqrt{\kappa}} \frac{\partial}{\partial t} [\sqrt{\kappa}\, \mathbf{D}] - \mathbf{H} = -\frac{4\pi}{c} \rho^* v. \tag{1.24.23}$$

In consequence, we can write Maxwell's equations (1.24.3) in "vector" form

$$\operatorname{curl} \mathbf{H} = \frac{1}{c\sqrt{\kappa}} \frac{\partial}{\partial t} [\sqrt{\kappa}\, \mathbf{D}] + \frac{4\pi}{c} \rho^* v,$$

$$\operatorname{div} \mathbf{D} = 4\pi\rho^*, \quad \operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c\sqrt{\kappa}} \frac{\partial}{\partial t} [\sqrt{\kappa}\, \mathbf{B}].$$

Here it is worth stressing that the vectors **E**, **D**, **B**, **H** introduced above do not feature the properties of four-vectors. This holds good even in the case of special relativity, which follows immediately from the transformation laws for the "vectors" **E** and **H** on passing from one inertial reference frame to another. The components of the tensor of the electromagnetic field F_{ik} are transformed by the tensor law. The components, according to the rules (1.24.18) and (1.24.19), are compared with the "vectors" **B** and **E**. This rule being noncovariant, the "vectors" **B** and **E**, as well as **H** and **D**, under transformations of reference frames are not changed by the tensor law, and therefore they are no four-vectors. Historically, however, it became more convenient to deal with these quantities than with the tensor of the electric field F_{ik}, i.e., to perform all calculations in terms of the "vectors" **E** and **H**, keeping in mind, however, that these "vectors" are not transformed by the tensor law in changing inertial reference frames.

Using the definitions (1.24.18) and (1.24.19), we will find the components T_0^0 of the tensor of energy-momentum of the electromagnetic field

$$T_0^0 = \frac{1}{4\pi}\left(-F_{0\alpha} F^{0\alpha} + \frac{1}{4} F_{lm} F^{lm}\right) = \frac{(E\mathbf{D}) + (\mathbf{BH})}{8\pi\sqrt{g_{00}}}. \tag{1.24.24}$$

This includes

$$E_{\alpha\beta\gamma} E^{\alpha\nu\mu} = (\delta_\beta^\nu \delta_\gamma^\mu - \delta_\beta^\mu \delta_\gamma^\nu).$$

The energy flux of the electromagnetic field is

$$P^\alpha = T^\alpha_0 = -\frac{1}{4\pi} F_{0\beta} F^{\alpha\beta} = -\frac{1}{4\pi} E_\beta F^{\alpha\beta}. \tag{1.24.25}$$

It follows from (1.24.19) that

$$F^{\alpha\beta} = -\frac{E^{\mu\alpha\beta}}{\sqrt{g_{00}}} H_\mu.$$

Therefore, the relation (1.24.25) can be written as

$$T^\alpha_0 = \frac{1}{4\pi\sqrt{g_{00}}} E^{\alpha\beta\mu} E_\beta H_\mu.$$

Since

$$([\mathbf{E}\,\mathbf{H}])^\alpha = E^{\alpha\beta\mu} E_\beta H_\mu,$$

the momentum vector of the electromagnetic field is

$$\mathbf{P} = \frac{1}{4\pi\sqrt{g_{00}}} [\mathbf{E}\mathbf{H}].$$

We now find the law of transformation of the coordinate "vectors" **E**, **D**, **H**, and **B** in passing from one inertial reference frame to another. We have the following relations:

$$E_1 = F_{01}, \quad E_2 = F_{02}, \quad E_3 = F_{03},$$

$$D^1 = -\sqrt{g_{00}}\, F^{01}, \quad D^2 = -\sqrt{g_{00}}\, F^{02}, \quad D^3 = -\sqrt{g_{00}}\, F^{03},$$

$$B^1 = -\frac{F_{12}}{\sqrt{\kappa}} = \frac{\sqrt{g_{00}}}{\sqrt{-g}} F_{23}, \quad B^2 = \frac{F_{13}}{\sqrt{\kappa}} = \frac{\sqrt{g_{00}}}{\sqrt{-g}} F_{13}, \tag{1.24.26}$$

$$B^3 = -\frac{F_{12}}{\sqrt{\kappa}} = -\frac{\sqrt{g_{00}}}{\sqrt{-g}} F_{12}, \quad H_1 = -\sqrt{-g}\, F^{23},$$

$$H_2 = \sqrt{-g}\, F^{13}, \quad H_3 = -\sqrt{-g}\, F^{12}.$$

Since the components of the coordinate tensor F_{ik} are transformed by

the tensor law

$$F^{\text{new}}_{ik} = \frac{\partial x^l_{\text{old}}}{\partial x^i_{\text{old}}} \frac{\partial x^m_{\text{old}}}{\partial x^k_{\text{new}}} F^{\text{old}}_{lm},$$

in passing from one inertial frame to another, the components of the coordinate "vector" of the electric field obey the following transformation law:

$$E^{\text{new}}_{\alpha} = F^{\text{new}}_{0\alpha} = \left(\frac{\partial t_{\text{old}}}{\partial t_{\text{new}}} \frac{\partial x^\beta_{\text{old}}}{\partial x^\alpha_{\text{new}}} - \frac{\partial x^\beta_{\text{old}}}{\partial t^\alpha_{\text{new}}} \frac{\partial t_{\text{old}}}{\partial x^\alpha_{\text{new}}}\right) E^{\text{old}}_\beta$$

$$+ \frac{1}{c} \frac{\partial x^\nu_{\text{old}}}{\partial t_{\text{new}}} \frac{\partial x^\beta_{\text{old}}}{\partial x^\alpha_{\text{new}}} F^{\text{old}}_{\nu\beta}. \tag{1.24.27}$$

Considering that

$$F_{\nu\beta} = -E_{\mu\nu\beta} B^\mu,$$

we find

$$E^{\text{new}}_{\alpha} = \left(\frac{\partial t_{\text{old}}}{\partial t_{\text{new}}} \frac{\partial x^\beta_{\text{old}}}{\partial x^\alpha_{\text{new}}} - \frac{\partial x^\beta_{\text{old}}}{\partial t_{\text{new}}} \frac{\partial t_{\text{old}}}{\partial x^\alpha_{\text{new}}}\right) E^{\text{old}}_\beta$$

$$- E_{\mu\nu\beta} B^\mu_{\text{old}} \frac{1}{c} \frac{\partial x^\nu_{\text{old}}}{\partial t_{\text{new}}} \frac{\partial x^\beta_{\text{old}}}{\partial x^\alpha_{\text{new}}}. \tag{1.24.28}$$

In a similar manner, the space components of the tensor of the electromagnetic field are

$$F^{\text{new}}_{\alpha\beta} = c\left(\frac{\partial t_{\text{old}}}{\partial x^\beta_{\text{new}}} \frac{\partial x^\nu_{\text{old}}}{\partial x^\alpha_{\text{new}}} - \frac{\partial t_{\text{old}}}{\partial x^\alpha_{\text{new}}} \frac{\partial x^\nu_{\text{old}}}{\partial x^\beta_{\text{new}}}\right) E^{\text{old}}_\nu$$

$$- E_{\mu\nu\eta} B^\eta_{\text{old}} \frac{\partial x^\mu_{\text{old}}}{\partial x^\beta_{\text{new}}} \frac{\partial x^\nu_{\text{old}}}{\partial x^\alpha_{\text{new}}}. \tag{1.24.29}$$

Since in passing from one inertial coordinate system to another the metric remains form-invariant, we have from (1.24.26)

$$B^1_{\text{new}} = -\frac{\sqrt{g_{00}}}{\sqrt{-g}} F^{\text{new}}_{23}, \quad B^2_{\text{new}} = \frac{\sqrt{g_{00}}}{\sqrt{-g}} F^{\text{new}}_{13},$$

$$B^3_{\text{new}} = -\frac{\sqrt{g_{00}}}{\sqrt{-g}} F^{\text{new}}_{12}. \tag{1.24.30}$$

Expressions (1.24.28), (1.24.29), and (1.24.30) thus define the law of transformation of the components of **B** and **E** in transitions from one inertial reference frame to another. The corresponding formulas for the transformation of the coordinate vectors **D** and **H** can also be readilv obtained.

We will now find the transformation law for the components of the physical vectors **B**, **H**, **E** and **D** in interframe passages. Note, above all, that the components of the physical vectors **D** and **E**, and also **B** and **H** coincide, if Maxwell's equations are considered in a vacuum ($\epsilon = \mu = 1$):

$$D^{\overline{\alpha}} = E_{\underline{\alpha}}, \quad B^{\overline{\alpha}} = H_{\underline{\alpha}}.$$

This is immediate from the relations (1.24.26). For instance, we have

$$D^{\overline{\alpha}} = -\sqrt{g_{\overline{0}\,\overline{0}}}\, F^{\overline{0}\,\overline{\alpha}}.$$

Since

$$\sqrt{g_{\overline{0}\,\overline{0}}} = 1, \quad F^{\overline{0}\,\overline{\alpha}} = \gamma^{\overline{0}\,\overline{0}}\gamma^{\overline{\alpha}\,\overline{\beta}} F_{\overline{0}\,\overline{\beta}} = -F_{\overline{0}\,\overline{\alpha}},$$

then, by (1.24.19),

$$D^{\overline{\alpha}} = E_{\underline{\alpha}}.$$

We find the relations between the components of the coordinate tensor and the components of the physically measurable vectors $E_{\underline{\alpha}}$ and $B^{\overline{\alpha}}$.

Using the expressions (1.23.9) for the components of the tetrad $\lambda^{\overline{n}}_{k}$, we will find the components $\lambda^{m}_{\underline{l}} = \gamma_{\underline{l}\,\underline{n}}\, g^{mk}\, \lambda^{\overline{n}}_{k}$ of the same tetrad:

$$\lambda^{1}_{\underline{1}} = -g^{11}\sqrt{\kappa_{11}}, \quad \lambda^{2}_{\underline{2}} = 1, \quad \lambda^{3}_{\underline{3}} = 1,$$

$$\lambda^{n}_{\underline{0}} = \frac{\delta^{n}_{0}}{\sqrt{g_{00}}}, \qquad \lambda^{0}_{\underline{1}} = -g^{01}\sqrt{\kappa_{11}}. \tag{1.24.31}$$

The components of the physical vector $E_{\underline{\alpha}}$ will then be

$$E_{\underline{1}} = F_{\underline{0}\,\underline{1}} = \lambda^{n}_{\underline{0}}\lambda^{m}_{\underline{1}} F_{nm} = -E_{1} g^{11}\sqrt{\frac{\kappa_{11}}{g_{00}}},$$

$$E_{\underline{2}} = \frac{E_2}{\sqrt{g_{00}}}, \quad E_{\underline{3}} = \frac{E_3}{\sqrt{g_{00}}}. \tag{1.24.32}$$

Likewise,

$$B_{\bar{1}} = \sqrt{g_{\bar{0}\bar{0}}}\, F_{\bar{2}\bar{3}} = F_{23},$$

$$B_{\bar{2}} = \sqrt{g_{\bar{0}\bar{0}}}\, F_{\bar{1}\bar{3}}$$

$$= -g^{11}\sqrt{\kappa_{11}}\, F_{13} - g^{01}\sqrt{\kappa_{11}}\, F_{03}, \quad (1.24.33)$$

$$B_{\bar{3}} = F_{\bar{2}\bar{1}} = (g^{11} F_{12} + g^{01} F_{02})\sqrt{\kappa_{11}}.$$

Using the transformation formulas (1.13.7) for passing from one inertial reference frame to another, we will find from relations (1.24.28) and (1.24.29)

$$E_1^{\text{new}} = E_1^{\text{old}}, \quad F_{23}^{\text{new}} = F_{23}^{\text{old}},$$

$$E_2^{\text{new}} = \left(\frac{c(c_2 - c_1) - V(c_1 + c_2)}{c(c_2 - c_1)} E_2^{\text{old}} - \frac{2c_1 c_2 V}{c(c_2 - c_1)} F_{12}^{\text{old}} \right)$$

$$\times \frac{1}{\sqrt{1 - V^2/c^2}},$$

$$E_3^{\text{new}} = \frac{[c(c_2 - c_1) - V(c_1 + c_2)]\, E_3^{\text{old}} - 2c_1 c_2 V F_{13}^{\text{old}}}{c(c_2 - c_1)\sqrt{1 - V^2/c^2}}, \quad (1.24.34)$$

$$F_{12}^{\text{new}} = \frac{2VE_2^{\text{old}} + [c(c_2 - c_1) + V(c_1 + c_2)]\, F_{12}^{\text{old}}}{c(c_2 - c_1)\sqrt{1 - V^2/c^2}},$$

$$F_{13}^{\text{new}} = \frac{2VE_3^{\text{old}} + [c(c_2 - c_1) + V(c_1 + c_2)]\, F_{13}^{\text{old}}}{c(c_2 - c_1)\sqrt{1 - V^2/c^2}},$$

where V is a physically measurable velocity of motion of one inertial reference frame relative to another.

We will also take into consideration the relation between the coordinate and physical components of the tensor F_{ik}

$$
\begin{aligned}
E_1 &= F_{01} = \lambda_0^{\bar n}\lambda_1^{\bar m} F_{\bar n \bar m} = \sqrt{g_{00}\,\kappa_{11}}\, F_{\bar 0 \bar 1}\,, \\
E_2 &= F_{02} = \lambda_0^{\bar n}\lambda_2^{\bar m} F_{\bar n \bar m} = \sqrt{g_{00}}\, F_{02}\,, \\
E_3 &= F_{03} = \sqrt{g_{00}}\, F_{\bar 0 \bar 3}\,, \qquad\qquad (1.24.35)\\
F_{12} &= \frac{g_{01}}{\sqrt{g_{00}}} F_{\bar 0 \bar 2} + \sqrt{\kappa_{11}}\, F_{\bar 1 \bar 2}\,, \quad F_{23} = F_{\bar 2 \bar 3}\,, \\
F_{13} &= \frac{g_{01}}{\sqrt{g_{00}}} F_{\bar 0 \bar 3} \sqrt{\kappa_{11}}\, F_{\bar 1 \bar 3}\,.
\end{aligned}
$$

From the first of (1.24.32) we then have

$$E^{\text{new}}_{\bar 1} - g^{11}\sqrt{\frac{\kappa_{11}}{g_{00}}}\, E^{\text{new}}_{1}.$$

Substituting into the right-hand side of this the first of (1.24.34) gives

$$E^{\text{new}}_{\bar 1} = -g^{11}\sqrt{\frac{\kappa_{11}}{g_{00}}}\, E^{\text{old}}_{1}\,.$$

But by virtue of the first of (1.24.35), we can write this expression as

$$E^{\text{new}}_{\bar 1} = -g^{11}\,\kappa_{11}\, E^{\text{old}}_{\bar 1}\,.$$

Taking into account expressions for the components of the tensors g_{ik} and $\kappa_{\alpha\beta}$, we have

$$E^{\text{new}}_{\bar 1} = E^{\text{old}}_{\bar 1}\,.$$

In much the same manner, we can find the transformation formulas for other components of the physical vectors **B** and **E**

$$
\begin{aligned}
&B^{\text{new}}_{\bar 1} = B^{\text{old}}_{\bar 1}\,, \\
&B^{\text{new}}_{\bar 2} = \frac{B^{\text{old}}_{\bar 2} - \frac{V}{c} E^{\text{old}}_{\bar 3}}{\sqrt{1 - V^2/c^2}}\,, \quad B^{\text{new}}_{\bar 3} = \frac{B^{\text{old}}_{\bar 3} + \frac{V}{c} E^{\text{old}}_{\bar 2}}{\sqrt{1 - V^2/c^2}}\,, \qquad (1.24.36)\\
&E^{\text{new}}_{\bar 2} = \frac{E^{\text{old}}_{\bar 2} + \frac{V}{c} B^{\text{old}}_{\bar 3}}{\sqrt{1 - V^2/c^2}}\,, \quad E^{\text{new}}_{\bar 3} = \frac{E^{\text{old}}_{\bar 3} - \frac{V}{c} B^{\text{old}}_{\bar 2}}{\sqrt{1 - V^2/c^2}}\,.
\end{aligned}
$$

We thus arrive at the well-known rules of transformation of three-dimensional components of the physical "vectors" of the strengths of the electric and magnetic fields for passages between inertial reference frames.

Obviously we can describe physical events in any permissible coordinate systems, but we should always remember that, along with coordinate treatment, there must be physical quantities, since only they are directly related to the structure of space-time, i.e., to length and time intervals. If we do remember this, a transition from coordinate quantities to physical ones will not change all well-known physical results. It is therefore clear that the description is independent of a coordinate system.

Chapter 2

GEOMETRY AND PHYSICS

All physical processes occur in space and time, therefore investigation into the geometry of space-time to reveal all of its properties plays an exceedingly important role. The connection between geometry and physics is especially manifest when determining natural geometry for this or that physical field, exploring the possibilities for deriving the conservation laws in theory, finding reference frames indistinguishable from some specified system in the course of any physical experiment. Answers to all these questions are largely dependent on the nature of a geometry, which enables us to give an unequivocal positive answer in some cases and negative in others. Hence the need to dwell on these issues.

Before we begin our treatment, however, let us recap the essential facts of tensor analysis and Riemannian geometry [22-24] which will be needed thereafter.

2.1. Tensor Analysis

Consider some set of n independent variables $x^i = [x^1, x^2, \ldots, x^n]$. This set can be viewed as a coordinate system in n-dimensional space in the sense that each system of values for these variables defines a point in space. We now define a system of n independent real-valued functions $f^k(x^i)$, $k = 1, \ldots, n$ of the variables x^i, which are continuous together with their partial derivatives up to order N. For these functions to be independent, it is necessary and sufficient for the Jacobian

$$J = \det \left\| \frac{\partial f^k}{\partial x^i} \right\| \tag{2.1.1}$$

to be nonzero. The collection of variables

$$x'^i = f^i(x^1, x^2, \ldots, x^n) \tag{2.1.2}$$

will then represent another coordinate system in space: if into the right-

hand side of (2.1.2) we substitute the coordinates x^i of a point A in space, these relations will give the coordinates x'^i of A in the new coordinate system.

Since the Jacobian (2.1.1) of the transformation (2.1.2) is nonzero at each point of space, the transformation (2.1.2) will be unique and reversible, i.e., in the neighborhood of each point the inverse transformation

$$x^i = \varphi^i(x'^1, x'^2, \ldots, x'^n) \tag{2.1.3}$$

is possible. Different physical processes are described using different ordered systems of functions $\Psi^a (a = 1, 2, \ldots, m)$ specified at each point of space or in some region of it, therefore we will have to explore the transformational properties of various systems of functions subjected to the coordinate transformations (2.1.2).

We will define the field of a geometric object (or simply a geometric object) of order P specified in an n-dimensional space (or in some region of it) as an ordered system of functions $\Psi^a(x^1, x^2, \ldots, x^n)$ of the coordinates of the space specified in each local coordinate system and varying under any transformation (2.1.2) by the law

$$\Psi'^a(x') = Y^a\left(x', \frac{\partial x'^i}{\partial x^{k_1}}, \frac{\partial^2 x'^i}{\partial x^{k_1}\partial x^{k_2}}, \ldots\right. \tag{2.1.4}$$

$$\left.\ldots, \frac{\partial^p x'^i}{\partial x^{k_1}\partial x^{k_2}\ldots\partial x^{k_p}}, \Psi^b(x(x'))\right),$$

$$(a, b = 1, 2, \ldots, m).$$

It is worth noting that all the quantities on the right-hand side of the transformation law of any geometric object (2.1.4) must be expressed, using the inverse transformation (2.1.3), as functions of the primed variables $x'^1, x'^2, \ldots, x'^n$. Geometric objects are classified according to the type of the transformation (2.1.4). The simplest geometrical object is the scalar field defined in each system of coordinates Ω and Ω' by a function $\Psi(x)$ or $\Psi'(x')$, respectively. When the coordinate system (2.1.2) is transformed, the scalar is transformed by the law

$$\Psi'(x') = \Psi(x(x')). \tag{2.1.5}$$

Expressions (2.1.2) and (2.1.5) make it possible to determine the transformation laws for the gradient of a scalar function $\partial\Psi(x)/\partial x^i$ and differentials dx^i. Indeed, on differentiating the right- and left-hand

sides of (2.1.5) with respect to x^i, we obtain

$$\frac{\partial \Psi'(x')}{\partial x'^i} = \frac{\partial}{\partial x'^i} \Psi(x(x')).$$

Using the rule of differentiation of composite functions

$$\frac{\partial}{\partial x'^i} \Psi(x(x')) = \frac{\partial \Psi(x)}{\partial x^k} \frac{\partial x^k}{\partial x'^i},$$

we arrive at

$$\frac{\partial \Psi'(x)}{\partial x'^i} = \frac{\partial \Psi(x)}{\partial x^k} \frac{\partial x^k}{\partial x'^i}. \tag{2.1.6}$$

Recall that hereinafter the same co- and contravariant indices imply summation. Likewise, on taking differentials of the right- and left-hand sides of (2.1.2), we have

$$dx'^i = \frac{\partial x'^i}{\partial x^k} dx^k. \tag{2.1.7}$$

We see thus that there exist at least two types of geometric objects of the first order with one index, which under coordinate transformations can vary by the laws (2.1.6) and (2.1.7), respectively. A system of functions that are transformed as a differential came to be known as a contravariant vector. In each coordinate system the contravariant vector $a^i(x)$ is defined by a set of n real-valued functions $a^i(x) = [a^1(x), a^2(x), \ldots, a^n(x)]$ taken in a certain order. It is transformed when passing to another coordinate system (2.1.2) by the law

$$a'^i(x') = \frac{\partial x'^i}{\partial x^k} a^k(x(x')). \tag{2.1.8}$$

The covariant vector $a_i(x)$ is also defined by a set on n real-valued functions $a_i(x) = [a_1(x), a_2(x), \ldots, a_n(x)]$ arranged in a certain order. But under the coordinate transformation (2.1.2) it is transformed like the gradient $\partial\Psi/\partial x^i$ of the scalar

$$a'_i = \frac{\partial x^k}{\partial x'^i} a_k(x(x')). \tag{2.1.9}$$

As compared with the vector, a more general geometrical object of the first order is the tensor. The k-order covariant and l-order contravariant tensor $T^{j_1 j_2 \ldots j_l}_{i_1 i_2 \ldots j_k}(x)$ is a geometrical object defined in each local coordinate system by a set of n^{l+k} functions taken in a certain order, which are transformed in passing to another coordinate system (2.1.2)

by the law

$$T'^{j_1 j_2 \dots j_l}_{i_1 i_2 \dots i_k}(x') = \frac{\partial x^{a_1}}{\partial x'^{i_1}} \frac{\partial x^{a_2}}{\partial x'^{i_2}} \cdots \frac{\partial x^{a_k}}{\partial x'^{i_k}} \frac{\partial x'^{j_1}}{\partial x^{b_1}} \frac{\partial x'^{j_2}}{\partial x^{b_2}} \cdots \cdots \frac{\partial x'^{j_l}}{\partial x^{b_l}} T^{b_1 b_2 \dots b_l}_{a_1 a_2 \dots a_k}(x(x')). \tag{2.1.10}$$

Obviously, the scalar (2.1.5), the contravariant (2.1.8) and covariant (2.1.9) vectors are special of the tensor at $k = 0$, $l = 0$; $k = 0$, $l = 1$, and $l = 0$, respectively.

A tensor for any number of co- and contravariant indices is said to be zero in a certain region of space, if all of its components are zero in that region. It follows from the transformational properties of the tensor (2.1.10) that in a certain region of space it becomes zero invariantly with respect to a choice of a coordinate system: if the tensor is zero in one coordinate system, it is zero another nonsingular coordinate system.

It should be emphasized that by the rules of differentiation of compound functions $x^i = x^i(x'(x''))$ and $x''^i = x''^i(x'(x))$ we get

$$\frac{\partial x^i}{\partial x''^\gamma} = \frac{\partial x^i}{\partial x'^l} \frac{\partial x'^l}{\partial x''^\gamma}, \quad \frac{\partial x''^i}{\partial x^\gamma} = \frac{\partial x''^i}{\partial x'^l} \frac{\partial x'^l}{\partial x^\gamma} \tag{2.1.11}$$

and the properties of the determinant

$$\det \left\| \frac{\partial x''}{\partial x} \right\| = \left\| \frac{\partial x''}{\partial x'} \right\| \text{et} \left\| \frac{\partial x'}{\partial x} \right\|$$

for transformations of tensors under coordinate transformations form a continuous group.

Before we leave the subject of the transformational properties of tensors, we will also note that the Kronecker symbol δ^i_k, which is a tensor, in any coordinate system has the following components

$$\delta^i_k = \begin{cases} 1 & \text{at } i = k, \\ 0 & \text{at } i \neq k. \end{cases} \tag{2.1.12}$$

Passing to tensor algebra, we will consider the basic invariant operations with tensors. There are four of them: addition, multiplication, convolution, and assigning of indices.

The operation of algebraic addition is applicable to tensors with the same number of co- and contravariant indices. To obtain an algebraic sum of two or more tensors in any coordinate system, the respective components of tensors are algebraically added up at each point

of space. In the special case of the addition of tensors of the second rank we have

$$S^{ik}(x) = A^{ik}(x) + B^{ik}(x). \tag{2.1.13}$$

Unlike addition, multiplication is valid for all kinds of tensors. For example, we will define the product of the tensor $A^{ik}(x)$ by the tensor $B^l_m(x)$ to be the tensor $c^{ikl}_m(x)$ whose components at each coordinate system are obtained by multiplying the respective components of the factors:

$$c^{ikl}_m(x) = A^{ik}(x) B^l_m(x). \tag{2.1.14}$$

The convolution operation is only applicable to a tensor that has both covariant and contravariant indices.

Convolution of the tensor $B^{\gamma_1 \gamma_2 \ldots \gamma_p \ldots \gamma_l}_{i_1 i_2 \ldots i_q\, i_k}$ by the pth contravariant and qth covariant indices is accomplished by multiplying the tensor by the Kronecker symbol $\delta^{i_q}_{\gamma_p}$, which reduced by unity the numbers of covariant and contravariant indices. In the special case of the mixed tensor of the second rank $B^i_k(x)$ a result of the convolution operation for indices is the scalar $B(x)$:

$$B(x) = B^i_k(x)\delta^k_i = B^i_i(x). \tag{2.1.15}$$

The operation of assigning indices makes it possible from any given tensor that has two or more identical indices to obtain a new tensor by changing the order of two or more indices in the writing of the original tensor. Since the definition of a tensor includes the order of its upper and lower indices, the operation will yield a tensor that will differ from the original one.

Suppose that we had tensor $A^{\gamma_1 \gamma_2}_{i_1 i_2}(x)$. After the assigning of indices we can obtain in the general case three different tensors

$$B^{\gamma_1 \gamma_2}_{i_1 i_2}(x) = A^{\gamma_2 \gamma_1}_{i_1 i_2}(x),$$

$$C^{\gamma_1 \gamma_2}_{i_1 i_2}(x) = .A^{\gamma_1 \gamma_2}_{i_2 i_1}(x),$$

$$Q^{\gamma_1 \gamma_2}_{i_1 i_2}(x) = A^{\gamma_2 \gamma_1}_{i_2 i_1}(x).$$

The assigning of indices is a composite part of the operations of symmetrization and alternation. Symmetrization in any N identical indices of the tensor is accomplished in the following manner. To begin with, $N!$ substitutions in these indices on the original tensor are performed.

Then the arithmetic mean of all the resultant $N!$ tensors is taken. The most common symmetrization is in two indices. The indices to be symmetrized are put in parentheses

$$
\begin{aligned}
A^{(j_1 j_2)\ldots j_l}_{i_1 i_2 \ldots i_k}(x) &= \frac{1}{2!}[A^{j_1 j_2 \ldots j_l}_{i_1 i_2 \ldots i_k}(x) + A^{j_2 j_1 \ldots j_l}_{i_1 i_2 \ldots i_k}(x)], \\
A^{j_1 j_2 \ldots j_l}_{(i_1 i_2)\ldots i_k}(x) &= \frac{1}{2!}[A^{j_1 j_2 \ldots j_l}_{i_1 i_2 \ldots i_k}(x) + A^{j_1 j_2 \ldots j_l}_{i_2 i_1 \ldots i_k}(x)].
\end{aligned} \tag{2.1.16}
$$

We will say that a tensor that remains unaltered by any substitution of several identical indices is symmetrical in these indices.

To carry out the operation of alternation in N identical indices of a tensor it is necessary to perform $N!$ various substitutions of these indices, to change the signs of odd substitutions, and to take the arithmetic mean of all the resultant $N!$ tensors. The indices to be alternated are then put in brackets

$$
\begin{aligned}
A^{[j_1 j_2]\ldots j_l}_{i_1 i_2 \ldots i_k}(x) &= \frac{1}{2!}[A^{j_1 j_2 \ldots j_l}_{i_1 i_2 \ldots i_k}(x) - A^{j_2 j_1 \ldots j_l}_{i_1 i_2 \ldots i_k}(x)], \\
A^{j_1 j_2 \ldots j_l}_{[i_1 i_2]\ldots i_k}(x) &= \frac{1}{2!}[A^{j_1 j_2 \ldots j_l}_{i_1 i_2 \ldots i_k}(x) - A^{j_1 j_2 \ldots j_l}_{i_2 i_1 \ldots i_k}(x)].
\end{aligned} \tag{2.1.17}
$$

2.2. Riemannian Geometry

A Riemannian space V_n is a real differentiable manifold M of class C^N (i.e., one with continuous partial derivatives with respect to all arguments up to order $N > 1$), at each point of which the field of the tensor

$$g_{ik} = g_{ik}(x^1, x^2, \ldots, x^n) \tag{2.2.1}$$

is defined that is, two times covariant, symmetric and nondegenerate:

$$g_{ik} = g_{ki}, \quad g = \det \| g_{ik} \| \neq 0. \tag{2.2.2}$$

The tensor $g_{ik} = g_{ik}(x^1, x^2, \ldots, x^n)$ will be referred to as the metric tensor of the Riemannian space V_n, whose determinant (2.2.2), by the rule of the multiplication of the determinants under the coordinate transformation (2.1.2), features the following transformational law: $g' = gJ^{-2}$.

It follows that the determinant of the metric tensor of a Riemannian space V_n is a scalar density of weight $+2$.

Using the metric tensor in a Riemannian space we can introduce the invariant differential form

$$ds^2 = g_{ik}\, dx^i dx^k, \tag{2.2.3}$$

called the basic metric form of this space (or an interval in V_n), and generalizing the concept of the square of the differential of an arc to include V_n. This allows us to give another, equivalent to the first one, definition of a space V_n: a Riemannian space V_n is a manifold M in which is specified the invariant quadratic differential form

$$ds^2 = g_{ik}\,dx^i dx^k,$$

where g_{ik} is a function of class C^N subject to the conditions

$$\det \| g_{ik} \| \neq 0, \quad g_{ik} = g_{ki}.$$

At each point of a Riemannian space V_n the expression (2.2.3) is an algebraic quadratic form with respect to the differentials dx^i. Under the coordinate transformation (2.1.2) the transformation of differentials dx^i at each point of V_n is a linear transformation with constant coefficients; therefore, at each fixed point in V_n we can apply the algebraic theory of transformations of quadratic forms. According to this theory, the metric tensor g_{ik}, which in this case is a matrix, can be diagonalized at that point. In the general case, the diagonal components of the matrix g_{ik} will not all be positive under real transformations (2.1.2). But by the law of inertia of quadratic forms the difference between the numbers of positive and negative diagonal components will be constant under any real transformation that diagonalises quadratic form. This difference is called the signature of the metric tensor of V_n at that point. It is to be noted that the name "signature of metric tensor" is often applied to the set of signs of diagonal components of the tensor at each point.

Therefore, in an arbitrary Riemannian space V_n the interval (2.2.3) will in the general case have a fixed sign. We will say that the interval (2.2.3) is (a) time-like, (b) isotropic, and (c) space-like, depending on which of the three cases below is realized:

$$\text{(a) } ds^2 > 0, \quad \text{(b) } ds^2 = 0, \quad \text{(c) } ds^2 < 0.$$

Since the determinant (2.2.2) of the covariant metric tensor (2.2.1) is nonzero at each point of the Riemannian space V_n, we can construct the contravariant metric tensor $g^{ki} = g^{kl}(x^1, x^2, \ldots, x^n)$ that is covariant to the metric tensor g_{ik}, i.e., a tensor subject to the conditions

$$g_{ik}\, g^{kl} = \delta^l_i. \tag{2.2.4}$$

The presence of contravariant and covariant metric tensors allows to define in V_n a number of new operations with tensors. Specifically, using the tensors g_{ik} and g^{kl} in V_n we introduce the operation of raising

the tensor index

$$A^p(x) = g^{mp} A_m(x). \tag{2.2.5}$$

In alike manner, we can define the operation of lowering the tensor index

$$A_p(x) = g_{mp} A^m(x). \tag{2.2.6}$$

These operations have a tensor nature; from the original tensor they produce a tensor with other numbers of covariant and contravariant indices.

Moreover, in a Riemannian space V_n the presence of a metric tensor allows to construct in a natural manner the tools of covariant (or, as sometimes said, absolute) differentiation. The central role in the definition of covariant differentiation is played by the notion of parallel translation of a tensor from an infinitesimally close point to a given point. The main linear part of the difference between the value of the tensor in the given point and its value obtained as a result of parallel translation from an infinitesimally close point will be the absolute differential which has a tensor nature in V_n. We will illustrate this by a simple example.

Let us consider some contravariant tensor of the first rank. Suppose that at some point x^i its value is A^i and at an infinitesimally close point $x^i + dx^i$ it is $A^i + dA^i$. Since dA^i is the difference of the components of the two tensors at infinitesimally close points, under coordinate transformations the quantity $dA^i = (\partial A^i/\partial x^l)dx^l$ is transformed not by the tensor law (2.1.10).

We will now translate in parallel the vector A^i from point x^i to an infinitesimally close point $x^i + dx^i$. As a result of the parallel translation the value of the vector at $x^i + dx^i$ will be $A^i + \delta A^i$. The increment δA^i in parallel translation must be linearly dependent on the vectors A^i, dx^i and the quantity Γ^i_{kl}, which all characterize the Riemannian space V_n

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l. \tag{2.2.7}$$

In this case, the result of the two successive infinitesimal parallel translations will coincide with the result of one parallel translation from the original point to the final one, and the result of the parallel translation of the sum of two vectors will be equal to the sum of the increments of each of them in this translation.

The difference of the vector $A^i + dA^i$ and the vector $A^i + \delta A^i$ that is translated from point x^i to point $x^i + dx^i$ will be

$$DA^i = dA^i - \delta A^i. \tag{2.2.8}$$

It can be shown that this difference is a tensor. From the expression (2.2.7) we have

$$DA^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl}A^k\right)dx^l. \tag{2.2.9}$$

Since the quantities DA^i and dx^l are tensors, the expression in parentheses on the right-hand side of (2.2.9) is a tensor as well. It is called the covariant derivative of A^i and denoted by

$$\nabla_l A^i = A^i_{;\,l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl}A^k. \tag{2.2.10}$$

In alike manner, we can construct an expression for the covariant derivative of the covariant vector B_i

$$\nabla_l B_i = \frac{\partial B_i}{\partial x^l} - \Gamma^k_{il} B_k. \tag{2.2.11}$$

The expression for the Christoffel symbols of the second kind Γ^i_{kl} (or, as they are sometimes referred to, connectedness of a Riemannian space) can be obtained by considering that a parallel translation does not change the magnitude of a scalar

$$\delta[g_{ik}A^iB^k] = 0.$$

After some simple calculation we obtain

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{ml}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^l} - \frac{\partial g_{kl}}{\partial x^m}\right). \tag{2.2.12}$$

It follows from the definition (2.2.7) that the Christoffel symbols of the second kind are no tensors. For the coordinate transformation (2.1.2) the connectedness components Γ^i_{kl} of a Riemannian space are transformed by the law

$$\Gamma'^i_{kl}(x') = \Gamma^p_{ms}(x(x'))\frac{\partial x'^i}{\partial x^p}\frac{\partial x^m}{\partial x'^k}\frac{\partial x^s}{\partial x'^l} + \frac{\partial x'^i}{\partial x^p}\frac{\partial^2 x^p}{\partial x'^k\partial x'^l}. \tag{2.2.13}$$

The connectedness of V_n is thus a linear unhomogeneous geometrical object of the second order.

We will also provide here some relations that will be useful for later treatment

$$\Gamma^i_{ki} = \frac{1}{2}g^{im}\frac{\partial g_{im}}{\partial x^k} = \frac{1}{2g}\frac{\partial g}{\partial x^k}, \tag{2.2.14}$$

$$g^k\Gamma^i_{kl} = -\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}[\sqrt{-g}\,g^{ik}], \tag{2.2.15}$$

which follows easily from the expression (2.2.12), if we take into consideration that

$$dg = gg^{ik}dg_{ik} = -gg_{ik}\,dg^{ik}. \tag{2.2.16}$$

The expressions (2.2.10), (2.2.11) for the covariant derivatives of the contravariant (A^i) and covariant (B_i) vectors can easily be generalized to the case of arbitrary tensor density.

The covariant derivative of the tensor density $A^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_l}(x)$ of weight w with respect to the coordinate x^m is defined as follows: to the conventional derivative of this density with respect to the coordinate x^m for each contravariant index i_s we add a term

$$\Gamma^{i_s}_{mp}\,A^{i_1 \ldots i_{s-1}\,p i_{s+1} \ldots i_k}_{j_1 j_2 \ldots j_l},$$

and for each covariant index j_s, the term

$$-\Gamma^{p}_{mj_s}\,A^{i_1 i_2 \ldots i_k}_{j_1 \ldots j_{s-1}\,p j_{s+1} \ldots j_l},$$

whereupon from the resultant expression we subtract the term

$$w\Gamma^s_{ms}A^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_l}.$$

We thus have

$$\begin{aligned}
&\nabla_m A^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_l}(x)\\
&= \frac{\partial}{\partial x^m}A^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_l}(x) + \Gamma^{i_1}_{mp}A^{p i_2 \ldots i_k}_{j_1 j_2 \ldots j_l}(x)\\
&+ \Gamma^{i_2}_{mp}A^{i_1 p \ldots i_k}_{j_1 j_2 \ldots j_l}(x) + \ldots + \Gamma^{i_k}_{mp}A^{i_1 i_2 \ldots p}_{j_1 j_2 \ldots j_l}(x)\\
&- [\Gamma^{p}_{mj_1}A^{i_1 i_2 \ldots i_k}_{p j_2 \ldots j_l}(x) + \Gamma^{p}_{mj_2}A^{i_1 i_2 \ldots i_k}_{j_1 p \ldots j_l}(x)\\
&\ldots + \Gamma^{p}_{mj_l}A^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_p}(x)]\\
&- w\Gamma^s_{ms}A^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_l}(x).
\end{aligned} \tag{2.2.17}$$

Consider some special cases of covariant differentiation of tensor densities.

(1) The covariant derivative of a scalar coincides with the partial derivative

$$\nabla_k \varphi = \frac{\partial \varphi}{\partial x^k}. \tag{2.2.18}$$

(2) The metric tensor of a Riemannian space V_n is covariantly constant:

$$\nabla_k g^{li} = 0, \qquad \nabla_k g_{li} = 0. \tag{2.2.19}$$

(3) The covariant divergence from the density of the contravariant vector A^i of weight +1 coincides with the "conventional" divergence

$$\nabla_i A^i = \frac{\partial A^i}{\partial x^i}. \tag{2.2.20}$$

(4) The covariant divergence from the density of the skew-symmetric tensor F^{ik} of weight +1 also coincides with the "conventional" divergence

$$\nabla_k F^{ki} = \frac{\partial F^{ki}}{\partial x^k}. \tag{2.2.21}$$

(5) The result of the alternation of the covariant derivative of the covariant vector A_k in a Riemannian space is independent of the metric

$$\nabla_i A_k - \nabla_k A_i = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}.$$

Likewise, the result of the alternation of a covariant derivative of a skew-symmetric, k time covariant ($k > 1$) tensor in Riemannian space V_n is also independent of the metric of the space

$$\nabla_{[j} A_{i_1 i_2 \ldots i_l]} = \frac{\partial}{\partial x^{[j}} A_{i_1 i_2 \ldots i_l]}.$$

(6) The covariant derivative of the Kronecker symbols is zero

$$\nabla_k \delta^i_l = 0.$$

(7) The determinant of the metric tensor is covariantly constant:

$$\nabla_l g = 0. \tag{2.2.22}$$

(8) The covariant differentiation of the sum, difference and product of tensors is accomplished according to the rules of conventional differentiation.

(9) The values of continuous second partial derivatives are independent of the order of differentiation

$$\frac{\partial^2}{\partial x^p \partial x^q} A^{j_1 j_2 \ldots j_k}_{i_1 i_2 \ldots i_l} = \frac{\partial^2}{\partial x^q \partial x^p} A^{j_1 j_2 \ldots j_k}_{i_1 i_2 \ldots i_l}$$

therefore, the result of covariant differentiation depends on the order of covariant derivatives

$$\nabla_p \nabla_q A^{j_1 j_2 \dots j_k}_{i_1 i_2 \dots i_l} = \nabla_q \nabla_p A^{j_1 j_2 \dots j_k}_{i_1 i_2 \dots i_l}$$
$$+ [A^{j_1 j_2 \dots j_k}_{s i_2 \dots i_l} R^s_{i_1 qp} + A^{j_1 j_2 \dots j_k}_{i_1 s \dots i_l} R^s_{i_2 qp} + \dots$$
$$\dots + A^{j_1 j_2 \dots j_k}_{i_1 i_2 \dots s} R^s_{i_l qp}] - [A^{s j_2 \dots j_k}_{i_1 i_2 \dots i_l} R^{j_1}_{sqp}$$
$$+ A^{j_1 s \dots j_k}_{i_1 i_2 \dots i_l} R^{j_2}_{sqp} + R^{jk}_{sqp} A^{j_1 j_2 \dots s}_{i_1 i_2 \dots i_l}], \tag{2.2.23}$$

where R^i_{klm} is the curvature tensor of V_n (the Riemann–Christoffel tensor)

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{ls}\Gamma^s_{km} - \Gamma^i_{ms}\Gamma^s_{kl}. \tag{2.2.24}$$

We discard the contravariant index in this expression and obtain the expression for the curvature tensor in terms of covariant components

$$R_{iklm} = \frac{1}{2}\left(\frac{\partial^2 g_{mi}}{\partial x^l \partial x^k} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m}\right.$$
$$\left. - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l}\right) + g_{qp}(\Gamma^p_{kl}\Gamma^q_{mi} - \Gamma^p_{km}\Gamma^q_{il}). \tag{2.2.25}$$

Because of its construction the curvature tensor (2.2.25) possesses the following symmetry properties:

(a) antisymmetry under exchange of indices in each of the pairs *ik* and *lm*:

$$R_{iklm} = -R_{kilm} = -R_{ikml} = R_{kiml}; \tag{2.2.26}$$

(b) symmetry under exchange of pairs of indices *ik* and *lm*:

$$R_{iklm} = R_{lmik}; \tag{2.2.27}$$

(c) equality to zero of the result of alternation in any three indices (the Ricci identity):

$$R_{[ikl]m} = \frac{1}{3!}(R_{iklm} + R_{likm} + R_{klim}) = 0. \tag{2.2.28}$$

Furthermore, the curvature tensor (26.24) obeys the Bianchi–Padov identity

$$R^i_{klm,p} + R^i_{kpl,m} + R^i_{kmp,l} = 0. \tag{2.2.29}$$

Thus, by (2.2.26)–(2.2.28), the curvature tensor of the Riemannian space (2.2.25) has

$$N = \frac{n^2(n^2 - 1)}{12} \tag{2.2.30}$$

independent components, in terms of which all the n^4 components of the tensor can be linearly expressed. Obtaining the convolution of indices i and l in the expression (26.24), we arrive at the symmetrical Ricci tensor

$$R_{km} = \frac{\partial \Gamma^i_{km}}{\partial x^i} - \frac{\partial \Gamma^i_{ki}}{\partial x^m} + \Gamma^i_{is}\Gamma^s_{km} - \Gamma^i_{ms}\Gamma^s_{ki}. \tag{2.2.31}$$

Raising one of the indices in the Ricci tensor (2.2.31) and obtaining its convolution with another one, we will get a scalar called the scalar curvature of a Riemannian space V_n

$$R = R^i_l = g^{im}R_{im}. \tag{2.2.32}$$

In Riemannian geometry an important role is played by the Hilbert tensor, which is a linear combination of the Ricci tensor and a scalar curvature

$$G^k_l = R^k_l - \frac{1}{2}\delta^k_l R. \tag{2.2.33}$$

It is easily seen that the raising of the index k in the Bianchi – Padov identity (2.2.29) and the subsequent convolution of the indices il and km in the resultant expression lead to the covariant equation of conservation for the Hilbert tensor (2.2.33)

$$\nabla_k G^k_l = 0. \tag{2.2.34}$$

The curvature tensor is one of the most important characteristics of a Riemannian space V_n, one that reflects the internal structure of the space.

In particular, the condition for the quadratic form (2.2.3) to be reducible to diagonal form at all points of V_n simultaneously is the requirement that the curvature tensor be equal to zero:

$$R^i_{klm} = 0. \tag{2.2.35}$$

The Riemannian space V_n is then said to be plain.

A more general Riemannian space V_n is a space of constant curvature: at each point of it the curvatures in the possible two-dimensional directions are the same.

For a space of constant curvature the Riemann – Christoffel tensor (2.2.25) has the form

$$R_{ipml} = k(g_{im}\, g_{pl} - g_{il} g_{pm}), \tag{2.2.36}$$

where k is the curvature.

By the Schur theorem in a space $V_n (n > 2)$ of constant curvature k remains the same at all points, i.e.,

$$k = \text{const.}$$

It follows from the expression (2.2.36) that $R = kn(n - 1)$, and therefore for $n > 1$ in a space V_n of constant curvature we have

$$R_{ipml} = \frac{R}{n(n-1)} (g_{im}\, g_{pl} - g_{il} g_{pm}), \tag{2.2.37}$$

where R is the scalar curvature.

Depending on the sign of R the following three cases are possible. If $R > 0$, the space is called a space of constant positive curvature (Riemann space). This space has a finite volume but has no boundaries. At $R = 0$ the space V_n is said to be plain (Euclidean or pseudo-Euclidean). If $R < 0$, then V_n is called a space of constant negative curvature, or hyperbolic space (Lobachevsky space). The last two spaces are infinite, with an infinite volume.

Let us now examine the behavior of the curvature tensor of a Riemannian space V_n at $n = 1, 2, 3$. If $n = 1$, a Riemannian space is unidimensional and each index in the expression (2.2.25) for the curvature tensor can only assume one value. Consequently, the curvature tensor of the space is identically zero, since we cannot construct even a single component of the space that would satisfy the conditions of skew-symmetry (2.2.26). The unidimensional Riemannian space V_1 is thus of necessity plain.

In the case of a two-dimensional Riemannian space V_2 the tensor indices can take on two values: 1 and 2. Therefore, the metric tensor g_{ik} will have three components: g_{11}, g_{12}, and g_{22}. The curvature tensor (2.2.25) in this case can have a nonzero component R_{1212}, and hence other components as well, which are obtained from it by substituting indices taking into consideration the symmetry properties (2.2.26)–(2.2.27). Since the Riemannian space V_2 only has one independent component of the curvature tensor, this tensor can be expressed through the scalar curvature R and the metric tensor g_{ik}. Indeed, considering that

$$R = 2R_{1212}[g^{11}g^{22} - (g^{12})^2],$$

we will get

$$R_{1212} = \frac{R}{2[g^{11}g^{22} - (g^{12})^2]}.$$

Since in a Riemannian space V_2 we have

$$[g^{11}g^{22} - (g^{12})^2]^{-1} = g = g_{11}g_{22} - (g_{12})^2,$$

we can write the component R_{1212} in the form

$$R_{1212} = \frac{R}{2}(g_{11}g_{22} - g_{12}g_{12}).$$

In V_2 we thus obtain

$$R_{iklm} = \frac{R}{2}(g_{il}g_{km} - g_{im}g_{lk}). \tag{2.2.38}$$

Accordingly, V_2 is of necessity a space of constant curvature. We can easily see that the Hilbert tensor (2.2.33) of this space is identically zero

$$G_k^i \equiv 0. \tag{2.2.39}$$

In a Riemannian space V_3 the curvature tensor has six independent components. The Ricci tensor and the metric tensor of the space have also six components each.

In complete analogy with the case of the Riemannian space V_2 we can show that the curvature tensor of V_3 can be expressed in terms of the Ricci tensor and the scalar curvature

$$R_{iklm} = -\frac{R}{2}(g_{il}g_{km} - g_{kl}g_{im})$$

$$+ (R_{il}g_{km} + R_{km}g_{il} - R_{kl}g_{im} - R_{im}g_{kl}). \tag{2.2.40}$$

In what follows we will only be interested in the Riemannian space V_4. The metric tensor g_{ik} and the Ricci tensor (2.2.31) of this space have ten independent components each, whereas the curvature tensor has twenty independent components. Therefore, in the Riemannian space V_4 the curvature tensor in the general case cannot be expressed through the scalar curvature (2.8.32) and the Ricci tensor (2.2.31).

As we know, all physical processes occur in space and time. These processes are often described using the four-dimensional space of events, whose points are elementary events. It is obvious that in the general case the space of events is a Riemannian space V_4. Experience shows, however, that one of the coordinates of this Riemannian space, namely

time, is distinguished. The coordinate is distinguished by the fact that after the quadratic form (2.2.3) is reduced throughout the space to diagonal form, the square of the time differential will have a sign opposite to the sign of the squared differentials of the other (space) coordinates. If we suppose for definiteness that the squared differential of time has the positive sign, we then obtain that the signature of the metric of the space of events is -2 and it has the form $(+, -, -, -)$.

To stress the fact that time is a distinguished component we will refer to the space of events as the pseudo-Riemannian space V_4 (or simply a Riemannian space-time), assuming time to be a zeroth coordinate of the space:

$$x^0 = ct.$$

If the curvature tensor of the pseudo-Riemannian space V_4 is equal to zero, this space is known as pseudo-Euclidean (or simply plain space-time). We will agree also that the Latin indices assume values 0, 1, 2, 3 and the Greek ones 1, 2, 3. We then have

$$x^i = [x^0, x^\alpha] = [ct, r].$$

It should be noted that in the pseudo-Riemannian space V_4 we may no longer carry out arbitrary coordinate transformations (2.1.2), but we must instead, because time is a distinguished coordinate, make use of permissible coordinate systems alone where the component g_{00} is positive

$$g_{00} > 0, \tag{2.2.41}$$

and the three-dimensional quadratic form $g_{\alpha\beta}dx^\alpha dx^\beta$ is negative, i.e.,

$$g_{\alpha\beta}dx^\alpha dx^\beta < 0. \tag{2.2.42}$$

Then, in any coordinate system meeting the conditions (2.2.41)-(2.2.42) the component x^0 will always have the nature of time, and the components x^α the nature of space coordinates.

By the Sylvester criterion, as we have seen above, the requirements (2.2.42) are equivalent to the conditions

$$g_{11} < 0, \qquad \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{vmatrix} < 0. \tag{2.2.43}$$

Note also that the condition for a coordinate system to be permissible imposes no constraints on the components $g_{0\alpha}$ of the metric tensor of Riemannian space-time.

2.3. Physical Field and Related Natural Geometry

In any physical theory where a field variable is a tensor quantity, the form of the differential field equations should not depend on the choice of coordinates in which this process is described. This can by achieved in two ways: by using in the field equations only covariant derivatives in the metric of space-time that is natural for this process, or by forming a tensor quantity out of field functions and their partial (noncovariant) derivatives. In the latter case the field equations will be substantially nonlinear.

Corresponding to any physical field is some geometry, called natural geometry, such that if there is no interaction with other fields the front of a free wave of this physical field moves along the geodesics of natural space-time.

The propagation of wave of massless field (the equation of characteristics) [9]

$$g^{ik} \frac{\partial \Psi}{\partial x^k} \frac{\partial \Psi}{\partial x^i} = 0, \tag{2.3.1}$$

as well as the motion of free material particles (the Hamilton – Jacobi equation)

$$g^{ik} \frac{\partial \Psi}{\partial x^i} \frac{\partial \Psi}{\partial x^k} = 1 \tag{2.3.2}$$

are defined by the metric tensor of the geometry that is natural for these processes.

The question of the choice of natural geometry is the question of which effective metric tensor effects the convolution of the higher derivatives in the Lagrangian density. A situation is quite possible, as it has been noted by Lobachevsky [25], where vatious physical phenomena will be described in terms of different natural geometries.

It follows equations (2.3.1) and (2.3.2) that the natural geometry of a physical theory permits of an experimental check drawing on the evidence concerning the motion of test particles and fields. Examination of the motion of test particles with mass and fields without mass enables us to determine the metric tensor of natural space-time up to a constant factor [5].

Studies of motions of a variety of forms of matter enables thus the nature of the geometry of the world's space-time to be checked experimentally.

The nature of the geometry of space-time largely predetermines the possibility of deriving the conservation laws for a closed system

of interacting fields. As we will see later, it is the requirements posed by the conservation laws for a closed system of physical fields that limits our choice of natural geometries to only three types of four-dimensional geometries.

The geometry of space-time also predetermines the existence or absence of a group of coordinate transformations that leave the metric tensor form-invariant, and hence the existence or absence of physically equivalent reference frames.

We will analyze these issues for the case of an n-dimensional Riemannian space V_n.

2.4. Form-Invariance Condition for Metric Tensor

As follows from Section 1.2, the condition for the form-invariance of a metric under the coordinate transformation (1.2.15) can be written as

$$g'_{ik}(x) = g_{ik}(x). \tag{2.4.1}$$

Let us take a closer look at it. Consider the infinitesimal coordinate transformation

$$x'^i = x^i + \xi^i(x) \tag{2.4.2}$$

with an infinitesimal vector $\xi^i(x)$, and define the conditions under which the equation (2.4.1) is obeyed in the first order in $\xi^i(x)$. In this case, the equation (2.4.1) can be written as

$$\delta_{\mathrm{L}} g_{ik} = g'_{ik}(x) - g_{ik}(x) = 0, \tag{2.4.3}$$

where we have introduced the notation δ_{L} for the variation in terms of the Lie differential (Lie variation). The condition for the metric of a Riemannian space-time (2.4.1) to be form-invariant under the infinitesimal coordinate transformation (2.4.2) is equivalent to the requirement that the Lie variation of the metric tensor becomes zero.

In the relation (2.4.3) the Lie variations of each of $n(n + 1)/2$ components of the metric tensor of a Riemannian space-time are not independent and they can be expressed through n components of the vector $\xi^i(x)$. Let us find this dependence. By the definition (2.4.3), we have

$$\delta_{\mathrm{L}} g^{ik} = g'^{ik}(x) - g^{ik}(x). \tag{2.4.4}$$

Under coordinate transformations the metric tensor of a Riemannian space-time exhibits the transformational law

$$g'^{ik}(x') = \frac{\partial x'^i}{\partial x^l}\frac{\partial x'^k}{\partial x^m} g^{ml}(x(x')).$$

And so for the transformations (2.4.2) we will obtain, up to terms linear in $\xi^i(x)$,

$$g'^{ik}(x+\xi) = g^{ik}(x) + g^{is}(x)\,\partial_s\xi^k + g^{ks}(x)\,\partial_s\xi^i.$$

Hence

$$g'^{ik}(x) = g^{ik}(x) + g^{is}(x)\partial_s\xi^k + g^{ks}(x)\partial_s\xi^i - \xi^s\partial_s g^{ik}(x).$$

Substituting this relation into expression (2.4.4), we will obtain

$$\delta_L g^{ik} = g^{is}\partial_s\xi^k + g^{ks}\partial_s\xi^i - \xi^s\partial_s g^{ik} = \nabla^i\xi^k + \nabla^k\xi^i. \tag{2.4.5}$$

By the relation

$$\delta_L g_{ik} = -g_{is}g_{km}\,\delta_L g^{sm}$$

we finally obtain, from (2.3) and (2.5),

$$\delta_L g_{ik} = -\nabla_i\xi_k - \nabla_k\xi_i = 0. \tag{2.4.6}$$

The form-invariance condition for the metric (2.4.3) will thus be met, if the equations (2.4.6) have a solution. In the literature equations (2.4.6) are known as the Killing equations, and the vector $\xi^i(x)$ that satisfy the equations (2.4.6), as the Killing vectors.

2.5. Space-Time Geometry and Conservation Laws

The Killing vectors play a fundamental role in physics. On the one hand, as we have seen, the existence of the Killing vectors in a Riemannian space-time guarantees the existence of a group of infinitesimal transformations of coordinates that leave the metric tensor form-invariant, and hence guarantee the availability of physically equivalent reference systems in which all physical phenomena occur identically under appropriate initial and boundary conditions.

On the other hand, the existence of the Killing vectors is closely linked with the existence of integral conservation laws for a closed system of interacting fields in a Riemannian space-time. It is well known [27] that a theory of any physical field can be constructed using the Lagrangian formalism. The physical field is then described by some function of coordinates and time, known as the field function. Equations for the function can be obtained from the variational principle of stationary action. In addition to the field equations the Lagrangian approach to constructing the classical theory of wave fields enables us to obtain a number of differential relations called the differential conservation laws. These relations are consequences of the action function being invariant under coordinate transformations of space-time; they

connect the local dynamic characteristics of the field with their covariant derivatives in a geometry that is natural to them.

In the current literature it is customary to distinguish two types of differential correlation laws – weak and strong.

The strong conservation law is generally defined as differential relation that holds due to the action function being invariant under coordinate transformations and that does not require that the equations of motion for the field be valid. On the other hand, weak conservation laws can be derived from the strong ones with account of the equations for a system of interacting fields. An example of a weak conservation law is the covariant conservation equation for the total tensor of energy-momentum of the system in an arbitrary Riemannian space-time

$$\nabla_m T^{ml} = \partial_m T^{ml} + \Gamma^m_{mi} T^{il} + \Gamma^l_{mi} T^{mi} = 0. \qquad (2.5.1)$$

This equation can be obtained as a consequence of the requirement that the action function of the system of interacting fields be invariant under an infinitesimal coordinate transformation provided that equations of motion hold for all fields.

It should be stressed that, despite their name, the differential conservation laws in the general case do not state that anything is conserved, locally or globally. These are simply differential identities relating various characteristics of a field, which hold because the action function does not change under an arbitrary coordinate transformation (i.e., it is a scalar). These relations owe their names to their analogy with the corresponding differential conservation laws in a pseudo-Euclidean space-time, in which integral laws can be obtained from differential conservation laws.

For instance, we can write a conservation law for the total tensor of the energy-momentum of a system of interacting fields (2.5.1) in a Cartesian coordinate system of a pseudo-Euclidean space-time

$$\frac{1}{c}\frac{\partial}{\partial t} T^{0i} = -\frac{\partial}{\partial x^\alpha} T^{i\alpha}.$$

Integrating this over some volume and using the Ostrogradsky–Gauss theorem, we obtain

$$\frac{1}{c}\frac{d}{dt}\int T^{0i} dV = -\oint T^{\alpha i} dS_\alpha.$$

This relation means that a change of the energy-momentum of a system of interacting fields in a volume equals the flux of energy-momen-

tum through the surface that bounds the volume. If there is no flux, i.e.,

$$\oint T^{\alpha i} dS_\alpha = 0,$$

we arrive at the conservation law for the total four-momentum of an isolated system

$$\frac{d}{dt} P^i = 0,$$

where

$$P^i = \frac{1}{c} \int T^{0i} dV.$$

Similar integral relations in a pseudo-Euclidean space-time can be obtained for the angular momentum as well.

On the other hand, if in a Riemannian space-time we have a differential covariant conservation equation this does not guarantee that we can deduce the corresponding integral conservation law.

Whether or not integral conservation laws can be obtained in a Riemannian space-time wholly depends on its geometry and the presence of the Killing vectors, or of the motion group in it. Let us take a closer look at the question, since the formalism developed here can be utilized to derive integral conservation laws also in arbitrary curvilinear coordinate systems of a pseudo-Euclidean space-time. We multiply the conservation equation (2.5.1) by the Killing vector, i.e., by the vector $\eta_l(x)$, that satisfies the Killing equations (2.4.6). Because the tensor T^{ml} is symmetrical, the expression obtained can be written as

$$\eta_l \nabla_m T^{ml} = \nabla_m [\eta_l T^{ml}] = 0.$$

Using the properties of the covariant derivative, we obtain from this

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^m} [\sqrt{-g}\, T^{ml} \eta_l] = 0.$$

Since the left-hand side of this is a scalar, we can multiply it by $\sqrt{-g}\, dV$ and integrate over some volume. As a result, we will obtain an integral conservation law in a Riemannian space-time

$$\frac{d}{dx^0} \int \sqrt{-g}\, T^{0l} \eta_l dV = -\oint \sqrt{-g}\, T^{\alpha l} \eta_l dS_\alpha. \tag{2.5.2}$$

If there is no flux of the three-dimensional vector $T^{\alpha l} \eta_l$ through the

surface that bounds the volume, we obtain

$$\int \sqrt{-g}\, T^{0l}\eta_l dV = \text{const}.$$

If there are Killing vectors we can thus obtain from the differential conservation law (2.5.1) the integral conservation laws (2.5.2).

2.6. Conditions for Killing Equations to be Solved

We now find out under what constraints on the metric of a Riemannian space-time the Killing equations (2.4.6) have solutions, i.e., under what conditions there exists a vector that obeys the equations (2.4.6).

The Killing equations (2.4.6) represent a system of linear partial differential equations of the first order. According to the general theory [22] we can reveal the conditions under which a system of partial differential equations can be integrated by reducing it to the form

$$\frac{\partial \Theta^a}{\partial x^i} = \Psi_i^a(\Theta^b, x^l), \tag{2.6.1}$$

where Θ^a are unknown functions; $i, l = 1, 2, \ldots, n; a, b = 1, 2, \ldots, M$. We can then find the conditions for the system (2.6.1) to be integrable from the relation

$$\frac{\partial^2 \Theta^a}{\partial x^i \partial x^l} = \frac{\partial^2 \Theta^a}{\partial x^l \partial x^i},$$

by replacing the first-order partial derivatives by the right-hand side of (2.6.1)

$$\frac{\partial \Psi_i^a}{\partial x^l} + \frac{\partial \Psi_i^a}{\partial \Theta^b} \Psi_l^b = \frac{\partial \Psi_l^a}{\partial x^i} + \frac{\partial \Psi_l^a}{\partial \Theta^b} \Psi_i^b. \tag{2.6.2}$$

If the conditions (2.6.2) are met identically from equations (2.6.1), then the system (2.6.1) is said to be completely integrable and its solution contains M parameters – a maximum of arbitrary constants for the given system.

If system (2.6.1) is not completely integrable, its solution will contain fewer arbitrary constants. We will determine under which conditions the solution to the Killing equations (2.4.6) in a Riemannian space V_n will contain the maximum possible number of parameters, and will find this number.

We will give our treatment in an explicitly covariant form, which is a covariant generalization of the above scheme of finding the integrability conditions for a system of partial differential equations. To accomplish this, we will first reduce the Killing equations (2.4.6)

to the required form. We differentiate covariantly the Killing equations (2.4.6) with respect to the variable x^l. As a result, we will get

$$\eta_{i;kl} + \eta_{k;il} = 0.$$

Hence

$$\eta_{i;jl} + \eta_{j;il} + \eta_{i;lj} + \eta_{l;ij} - \eta_{j;li} - \eta_{l;ji} = 0.$$

Rearranging, we obtain

$$\eta_{i;jl} + \eta_{i;lj} + [\eta_{j;il} - \eta_{j;li}] + [\eta_{l;ij} - \eta_{l;ji}] = 0. \tag{2.6.3}$$

On the other hand, the commutation rules for covariant derivatives dictate that

$$\eta_{i;lj} - \eta_{i;jl} = \eta_h R^h_{ilj}. \tag{2.6.4}$$

Substituting the expression (2.6.4) into (2.6.3) gives

$$2\eta_{i;jl} + \eta_h R^h_{ilj} + \eta_h R^h_{jil} + \eta_h R^h_{lij} = 0. \tag{2.6.5}$$

Using the Ricci identity

$$R^h_{ijl} + R^h_{jli} + R^h_{lij} = 0, \tag{2.6.6}$$

we get

$$\eta_h R^h_{ilj} + \eta_h R^h_{jil} = \eta_h R^h_{lij}.$$

Therefore, we can rewrite (2.6.5) as

$$\eta_{i;jl} = -\eta_h R^h_{lij}.$$

We thus obtain the following covariant equations

$$\eta_{i;j} + \eta_{j;i} = 0, \quad \eta_{i;jl} = -\eta_h R^h_{lij}. \tag{2.6.7}$$

We transform this system of covariant differential equations into a system that only contains first covariant derivatives. So we introduce, in addition to n unknown components of η_i, an unknown tensor λ_{ij} by

$$\eta_{i;j} = \lambda_{ij}. \tag{2.6.8}$$

This tensor contains n^2 unknown components, but only $n(n-1)/2$ of these are independent, since the tensor is skew-symmetric by (2.4.6) and (2.6.8):

$$\lambda_{ij} + \lambda_{ji} = 0. \tag{2.6.9}$$

From the foregoing the desired system of covariant differential equations becomes

$$\begin{aligned} \eta_{i;j} &= \lambda_{ij}, \\ \lambda_{ij;l} &= \eta_h R^h_{lji}. \end{aligned} \tag{2.6.10}$$

We have thus reduced the Killing equations (2.4.6) to a system of special form; it consists of linear differential equations solvable for first-order covariant derivatives.

The system is a covariant generalization of system (2.6.1), the role of unknown functions Θ^α being played here by $n(n+1)/2$ components of the tensors λ_{ij} and η_i

$$\Theta^\alpha = [\eta_i, \lambda_{ij}].$$

The condition for system (2.6.10) to be integrable can be obtained from the commutation rule for covariant derivatives, which is a consequence of the independence of the order of derivatives in partial differentiation. From this rule we have

$$\begin{aligned} &\eta_{i;jl} - \eta_{i;lj} = \eta_h R^h_{ijl}, \\ &\lambda_{im;l} - \lambda_{im;lj} = \lambda_{ih} R^h_{mjl} + \lambda_{hm} R^h_{ijl}. \end{aligned} \tag{2.6.11}$$

Substituting expressions (2.6.10) for the first covariant derivatives on the left of these, and using the property (2.6.9) that the tensor λ_{im} is antisymmetric, we will obtain the integration conditions for (2.6.10) in the form

$$\lambda_{im;j} - \lambda_{ij;m} = \eta_h R^h_{imj}, \tag{2.6.12}$$

$$[\eta_h R^h_{jmi}]_{;l} - [\eta_h R^h_{lmi}]_{;j} = \lambda_{ih} R^h_{mjl} + \lambda_{hm} R^h_{ijl}. \tag{2.6.13}$$

It is easily seen that the first of these expressions holds identically by virtue of the equations (2.6.10) of the system and of the properties of the curvature tensor. And so if the condition (2.6.13) will identically hold only due to the symmetry of a Riemannian space-time, system (2.6.10) will be completely integrable and hence the Killing equations (2.4.6) will contain the greatest possible number $M = n(n+1)/2$ of arbitrary constants. Since the unknown functions η_i and $\lambda_{im} = -\lambda_{mi}$ in system (2.6.10) must be independent, the left-hand side of (2.6.13) becomes identically zero only subject to the conditions

$$R^h_{mij;l} - R^h_{lij;m} = 0, \tag{2.6.14}$$

$$\begin{aligned} &\delta^s_j R^h_{iml} - \delta^h_j R^s_{iml} - \delta^s_i R^h_{jml} + \delta^h_i R^s_{jml} + \delta^s_l R^h_{mij} - \delta^h_l R^s_{mij} - \\ &- \delta^s_m R^h_{lij} + \delta^h_m R^s_{lij} = 0. \end{aligned} \tag{2.6.15}$$

Convolution of (2.6.15) in indices f and s, using the relations

$$R^s_{ims} = -R_{im}, \quad R^s_{smi} = 0$$

and the Ricci identity (2.6.6), gives

$$-(n-1)R^{h}_{mij} = \delta^h_j R_{mi} - \delta^h_i R_{jm}.$$

It follows that

$$-R_{lmij} = \frac{1}{(n-1)}\,[g_{jl}R_{mi} - g_{il}R_{jm}]. \qquad (2.6.16)$$

Multiplying this by g^{mi} gives

$$nR_{jl} = g_{jl}R.$$

Substituting this into (2.2.16), we will obtain the condition for (2.6.15) to hold identically

$$R_{lmij} = \frac{R}{n(n-1)}\,[g_{jl}g_{mi} - g_{il}g_{jm}]. \qquad (2.6.17)$$

From (2.6.17) and (2.6.14) we derive the requirement to be met by the scalar curvature

$$[\delta^h_j g_{im} - \delta^h_i g_{jm}]\,\frac{\partial}{\partial x^l}\,R - [\delta^h_j g_{li} - \delta^h_i g_{lj}]\,\frac{\partial}{\partial x^m}\,R = 0.$$

Multiplying this by $\delta^l_h g^{mi}$, we will have

$$(n-1)\,\frac{\partial}{\partial x^i}\,R = 0.$$

Since in the case under consideration $n > 1$, for the condition to be met it is necessary and sufficient that R = const. Accordingly, the integration conditions (2.6.14) and (2.6.15) for the Killing equations (2.4.6) will hold identically, if and only if the curvature tensor for a Riemannian space-time has the form

$$R_{lmij} = \frac{R}{n(n-1)}\,[g_{jl}g_{mi} - g_{il}g_{jm}],$$

where R = const.

In consequence, the Killing equations have solutions with the greatest possible number $M = n(n+1)/2$ of arbitrary constants (parameters) only when a Riemannian space V_n is a space with constant curvature. If then V_n is not a space of constant curvature the number of parameters will be less.

This suggests that on the mathematical side the presence of integral conservation laws for energy-momentum and angular momentum and the existence of a group of physically equivalent reference

frames are reflections of definite properties of space-time: namely of its homogeneity and isotropicity. There exist three types of four-dimensional spaces [24] that feature the properties of homogeneity and isotropicity to such an extent that they permit introducing ten integrals of motion for a closed system and the ten-parameter group of coordinate transformations under which the metric tensor remains form-invariant: the space with a constant negative curvature (Lobachevsky space), the space of zero curvature (pseudo-Euclidean space) and the space of constant positive curvature (Riemannian space). The first two spaces are infinite, they have infinite volumes; the third space is closed, it has a finite volume but has no boundaries.

2.7. Killing Vectors and Conservation Laws in Pseudo-Euclidean Space-Time

We now find the Killing vectors in an arbitrary curvilinear coordinate system of a pseudo-Euclidean space-time. To begin with, we write the Killing equations in the Cartesian coordinate system

$$\frac{\partial}{\partial x^i}\eta_j + \frac{\partial}{\partial x^j}\eta_i = 0.$$

We thus find the Killing vectors from a system of ten linear partial differential equations of the first order.

Solving the system, following conventional rules, yields

$$\eta_i = a_i + w_{ij}x^j, \tag{2.7.1}$$

where a_i is an arbitrary constant infinitesimal vector; w_{ij} is an arbitrary constant infinitesimal vector subject

$$w_{ij} = -w_{ji}. \tag{2.7.2}$$

As it should be expected, solution (2.7.1) thus contains all the ten arbitrary parameters.

Since expression (2.7.1) includes ten independent parameters, we have virtually ten independent Killing vectors, and the relation (2.7.1) is a linear combination of these ten independent vectors.

What is the meaning of these parameters? Substituting (2.7.2) into (2.4.2), we will get

$$x'^i = x^i + a^i + w^i_j x^j. \tag{2.7.3}$$

This suggests that the four parameters a^i are the components of the four-vector of infinitesimal translations of the reference frame. The three parameters $w_{\alpha\beta}$ are the components of the tensor of rotation through an infinitesimal angle about some axis (the so-called

pure rotations). The three parameters $w_{0\beta}$ describe infinitesimal rotations in the plane $x^0 x^\beta$, called the Lorentz rotations. The metric tensor γ_{mi} is form-invariant under translations, therefore a pseudo-Euclidean space-time is homogeneous; its properties are independent of where in the space-time the origin of coordinates is located. Likewise, the form-invariance of the metric tensor γ_{mi} under four-dimensional rotations betokens its isotropicity. It follows that in a pseudo-Euclidean space-time all directions are identical.

We conclude that a speudo-Euclidean space-time allows a ten-parameter group of motions which includes a four-parameter subgroup of translations and a six-parameter subgroup of rotations. The presence of this group of motions and the existence of the corresponding Killing vectors guarantees the presence of ten integral conservation laws for energy-momentum and angular momentum for a system of interacting fields.

Indeed, considering that in a Cartesian coordinate system $\sqrt{-g} = 1$, we will from (2.5.2) obtain for the translation subgroup ($\eta_i = a_i$)

$$\frac{d}{dx^0} \int T^{0i} a_i dV = - \oint T^{\alpha i} a_i dS_\alpha .$$

Since a^i is an arbitrary constant vector, we have from this relation

$$\frac{d}{dx^0} \int T^{0i} dV = - \oint T^{\alpha i} dS_\alpha .$$

For an isolated system of interacting fields the expression on the right-hand side of this relation is zero, and so its total four-momentum is conserved, i.e.

$$P^i = \frac{1}{c} \int T^{0i} dV = \text{const.} \tag{2.7.4}$$

Likewise, at

$$\eta_i = w_{ij} x^j$$

we will get

$$\frac{d}{dx^0} \int T^{0i} x^j w_{ij} dV = - \oint T^{\alpha i} x^j w_{ij} dS_\alpha .$$

Since the constant vector w_{ij} is skew-symmetric, we have the integral law of angular momentum conservation .

$$\frac{d}{dx^0} \int [T^{0i} x^j - T^{0j} x^i]\, dV = - \oint [T^{\alpha i} x^j - T^{\alpha j} x^i]\, dS_\alpha . \tag{2.7.5}$$

The total angular momentum of an isolated system is conserved, because the right-hand side of (2.7.5) vanishes:

$$M^{ij} = \frac{1}{c} \int [T^{0i}x^j - T^{0j}x^i]\, dV = \text{const.} \tag{2.7.6}$$

It should be noted that in arbitrary curvilinear coordinates of a pseudo-Euclidean space-time, since the quantities x^i and η^i are tensors, we can solve the Killing equations (2.4.6) from the solution (2.7.1) of these equations in a Cartesian coordinate system. To do so, we will in (2.7.1) change from the Cartesian coordinates x^i to arbitrary curvilinear coordinates x^i

$$x^i = f^i(x^l_{\text{new}}).$$

Then

$$\eta_i^{\text{new}} = \frac{\partial f^l}{\partial x^i_{\text{new}}}\, \eta_l(x_{\text{old}}(x_{\text{new}})).$$

Therefore, in an arbitrary curvilinear coordinate system of a pseudo-Euclidean space-time the Killing vectors have the form

$$\eta_i = \frac{\partial f^l}{\partial x^i_{\text{new}}}\, a_l + \frac{\partial f^l}{\partial x^i_{\text{new}}}\, f^j(x_{\text{new}})\, w_{lj}. \tag{2.7.7}$$

Generalization of the expressions (2.7.4)–(2.7.6) to the case of arbitrary curvilinear coordinates is a straightforward exercise. Following along the same lines as before, we find the four-momentum for an isolated system

$$P^i = \frac{1}{c} \int \sqrt{-g(x_{\text{new}})}\, dx^1_{\text{new}} dx^2_{\text{new}} dx^3_{\text{new}}\, T^{0l}(x_{\text{new}}) \frac{\partial f^i(x_{\text{new}})}{\partial x^l_{\text{new}}}.$$

The skew-symmetric tensor of the angular momentum then becomes

$$M^{ij} = \frac{1}{c} \int dx^1_{\text{new}} dx^2_{\text{new}} dx^3_{\text{new}} \sqrt{-g(x_{\text{new}})}\, T^{0s}(x_{\text{new}})$$

$$\times \left[f^j(x_{\text{new}}) \frac{\partial f^i(x_{\text{new}})}{\partial x^s_{\text{new}}} - f^i(x_{\text{new}}) \frac{\partial f^j(x_{\text{new}})}{\partial x^s_{\text{new}}} \right].$$

The nature of the geometry of space-time thus determines whether we can obtain the integral conservation laws and whether there exist physically equivalent reference frames. In four dimensions, including physical space-time, only spaces with constant curvature possess all

the ten conservation laws and the ten-parameter group of physically equivalent systems; in other spaces there are fewer laws.

It is worth noting that, beginning with the work of Poincaré [5], in the literature the question of the connection between geometry and physics has been discussed. It is maintained that since it is only a marriage of geometry and physics that yields things that can be tested experimentally, a geometry to describe phenomena can be taken arbitrary, although it may happen to complicate the description. As we have seen, however, the choice of geometry is no matter of convention, it is rather determined by physical principles, namely by the laws of conservation of energy-momentum and angular momentum for a closed system of interacting fields. And since these physical laws are daily tested in studies of the properties of matter, thereby the geometry of space-time of the real world is tested.

The above treatment has indicated therefore that electrodynamics has actually led to the discovery of the unity of space-time. As a result of the discovery the principle of relativity has lost its fundamental significance in theory and became a consequence of the pseudo-Euclidean geometry of unified space-time.

At the same time, the pseudo-Euclidean geometry of space-time enabled us to introduce a new principle – the generalized principle of relativity, which is valid both for inertial and noninertial reference frames. This circumstance clearly indicates that the opinion, so common in the literature [8, 11, 19, 64], that the special theory of relativity cannot be used to describe physical processes in noninertial reference frames is erroneous.

2.8. Riemannian Geometry and Gravitation

Among the interactions possible in nature gravitation stands apart. First, this interaction is universal, it affects all the forms of matter known so far. Second, its action on matter is fundamentally different from the action of other forces: as evidenced by the measurements of the deflection of the sunbeam and the lagging of a radio signal in the gravitational field of the Sun, a gravitational field acts on a wave by distorting its front. The propagation of the wave front of a massless field (the equation of characteristics)

$$g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} = 0$$

and the motion of free particles (the Hamilton – Jacobi equation)

$$g^{ik} \frac{\partial \Psi'}{\partial x^i} \frac{\partial \Psi}{\partial x^k} = m^2 c^2$$

is determined by the metric tensor. It follows that a gravitational field sort of changes the geometry for other fields of matter. It may well be asked, therefore, whether we can introduce a gravitational field so that its action of the other fields of matter would be equivalent to the change of geometry for these fields, and the gravitational field itself could be regarded as a field *à la* Faraday – Maxwell, with its conventional property of a carrier of energy-momentum.

An answer to this question is to be found in [28-30]. Later in the book we will follow these works. As one is led to conclude from the results of Sections 2.5 and 2.6, we can drive integral conservation laws in any physical theory largely because there exist the Killing vectors of space-time, or of the group of motions a space (metric). There only exist three types of four-dimensional spaces that are so homogeneous and isotropic that we can obtain all the ten integral conservation laws for a closed system: the space of constant negative curvature (Lobachevsky space), the space of zero curvature (pseudo-Euclidean space), and the space of constant positive curvature (Riemannian space).

Since experimental data on strong, electromagnetic and weak interactions show that for fields associated with these interactions in the absence of a gravitational field the geometry of space-time is pseudo-Euclidean, we can venture the hypothesis that this geometry is common for all physical processes, including gravitational ones. The validity of energy-momentum conservation for a closed system, irrespective of angular momentum conservation, among the three geometries with constant curvature brings out the zero-curvature geometry of space-time (pseudo-Euclidean space).

A further key element of the field approach we now attempt to expound is the principle of identity (of geometrization), which states that the equations of motion of matter under the action of a gravitational field in a pseudo-Euclidean space-time with the metric tensor γ_{ik} can be identically represented as the equations of motion of matter in a certain effective Riemannian space-time with the metric tensor g_{ik}, which depends on the gravitational field and the metric tensor γ_{ik}. This principle thus defines, on the one hand, the equivalence of these two ways of describing the motion of matter under a gravitational field; on the other hand, it determines the character of interaction of the gravitational field with matter and corresponds to a certain choice of Lagrangian interaction of the gravitational field with matter. It should be emphasized that the field approach to the theory of gravitational interaction ignores the nature of the gravitational field. We do not know the nature of the field, only time and new experimental evidence yield an answer to this question. One possible realization of the approach

consists in using the symmetric tensor field of the second rank as a gravitational field.

We now turn to the character of the conservation laws for all such local theories of gravity, without specifically selecting the density of the Lagrangian. Relying on the basic principles of the field approach, we can write the density of the Lagrangian of a system consisting of matter and a gravitational field in the form

$$\mathfrak{L} = \mathfrak{L}_g(\gamma_{ik}, \varphi_{ik}) + \mathfrak{L}_M(g_{ik}, \varphi_A), \tag{2.8.1}$$

where γ_{ik} is the metric tensor of the pseudo-Euclidean space-time, g_{ik} is the metric tensor of the effective Riemannian space-time, φ_{ik} is the gravitational field, φ_A is other fields of the matter. Without loss of generality, we will consider that the metric tensor g_{ik} is a local function depending on the metric tensor γ_{ik} of the gravitational field φ_{ik} and their partial derivatives up to the second order:

$$g_{lm} = g_{lm}(\gamma_{ik}, \partial_s \gamma_{ik}, \partial_{st}\gamma_{ik}, \varphi_{ik}, \partial_s \varphi_{ik}, \partial_{st}\varphi_{ik}, \ldots$$
$$\ldots, \gamma^{ik}, \partial_s \gamma^{ik}, \partial_{st}\gamma^{ik}). \tag{2.8.2}$$

We will consider that the density of the matter Lagrangian is dependent on the metric tensor g_{ik} and the matter fields $\mathfrak{L}_A$, the latter entering into the Lagrangian density with the derivatives of order not higher than the first. It is easily seen that the matter Lagrangian density will include the partial derivatives of the gravitational field up to the second order. The density of the Lagrangian of the gravitational field $\mathfrak{L}_g'$ will be taken to be dependent on the metric tensor γ_{ik}, the gravitational field and their partial derivatives through the third order. To derive the conservation laws we will, as in Section 1.9, use the covariant method of infinitesimal displacements. Since the action S is a scalar for an infinitesimal coordinate transformation

$$x'^i = x^i + \xi^i(x) \tag{2.8.3}$$

the variations of the matter action δS_M and of the gravitational field δS_g will be zero. Since the density of the matter Lagrangian includes both covariant and contravariant components of the metric tensor, we will vary the density of the Lagrangian in them as the independents, and then we will take into account the relation between their variations

$$\delta g^{lm} = -g^{im} g^{lm} \delta g_{in}.$$

We will use the same treatment when varying in the components γ_{ik} and γ^{lm} of the metric tensor of plane space-time.

Reasoning along the same lines as in Section 1.9, we will write the variation of the integral of the matter action under the transformation

(2.8.3) in the form

$$\delta S_{\mathrm{M}} = \frac{1}{c} \int d^4x \left(\frac{\Delta \mathfrak{L}_{\mathrm{M}}}{\Delta g_{in}} \delta_{\mathrm{L}} g_{in} + \frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} \delta_{\mathrm{L}} \varphi_A + \mathrm{Div} \right) = 0, \quad (2.8.4)$$

where Div stands for the divergent terms, which are of no significance for our treatment. We will now find the Lie variations that enter into this expression. By definition, we have

$$\delta_{\mathrm{L}} g_{in} = \delta_{\mathrm{c}} g_{in} - \xi^l \partial_l g_{in}.$$

Since the metric tensor of the effective Riemannian space-time, as any tensor of the second rank, obeys the transformational law

$$g'_{in}(x + \xi) = \frac{\partial x^q}{\partial x'^i} \frac{\partial x^p}{\partial x'^n} g_{pq}(x),$$

we can readily find

$$g'_{in}(x + \xi) = g_{in}(x) - g_{il}\, \partial_n \xi^l - g_{nl}\, \partial_i \xi^l.$$

Therefore, the coordinate variation δ_{c} of the metric tensor g_{in} will be

$$\delta_{\mathrm{c}} g_{in} = g'_{in}(x + \xi) - g_{in}(x) = - g_{il}\, \partial_n \xi^l - g_{nl}\, \partial_i \xi^l.$$

Hence

$$\delta_{\mathrm{L}} g_{in} = - g_{il}\, \partial_n \xi^l - g_{nl}\, \partial_i \xi^l - \xi^l \partial_l g_{in}$$

or in covariant form

$$\delta_{\mathrm{L}} g_{in} = - g_{il} D_n \xi^l - g_{nl} D_i \xi^l - \xi^l D_l g_{in}. \quad (2.8.5)$$

Since the geometric nature of matter fields φ_A is not specified beforehand, we will write the Lie variation of them in the general form

$$\delta_{\mathrm{L}} \varphi_A = - \xi^l D_l \varphi_A + F_{A;l}^{B;n} \varphi_B D_n \xi^l, \quad (2.8.6)$$

where the tensor F can assume one form or another depending on the geometric type of the field φ_A. Introducing the notation

$$T^{ik} = -2 \frac{\Delta \mathfrak{L}_{\mathrm{M}}}{\Delta g_{ik}} = -2 \left[\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta g_{ik}} - g^{is} g^{kp} \frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta g^{ps}} \right]$$

for the energy-momentum tensor of matter in a Riemannian space-time and substituting the relations (2.8.5) and (2.8.6) into the ex-

pression (2.8.4), we will obtain after some rearrangement

$$\delta S_{\mathrm{M}} = \frac{1}{c} \int dx \left(- \xi^l \left[D_n (g_{il} T^{in}) - \frac{1}{2} T^{in} D_l g_{in} \right.\right.$$

$$\left.\left. + \frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} D_l \varphi_A + D_n \left(\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} F^{B;\,n}_{A;\,l} \varphi_B \right) \right] + \mathrm{Div} \right) = 0.$$

Since the vector ξ^l and its derivatives in this expression are arbitrary, we can form this derive the following strong conservation law:

$$D_n (g_{il} T^{in}) - \frac{1}{2} T^{in} D_l g_{in} =$$

$$= - \frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} D_l \varphi_A - D_n \left(\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} F^{B;\,n}_{A;\,l} \varphi_B \right).$$

We now express the covariant derivatives on the left side of the identity through the partial derivatives and connectedness of the plain space-time γ^l_{ip}. Considering that T^{in} is the density of a tensor of weight +1, we will have

$$\partial_n (g_{il} T^{in}) - \frac{1}{2} T^{in} \partial_l g_{in} = - \frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} D_l \varphi_A - D_n \left(\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} F^{B;\,n}_{A;\,l} \varphi_B \right).$$

Expressing $\partial_l g_{in}$ through the connectednesses of the effective Riemannian space-time $\partial_l g_{in} = - g_{is} \Gamma^s_{ln} - g_{ns} \Gamma^s_{li}$, we will get

$$g_{il} \nabla_n T^{in} = - \frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} D_l \varphi_A - D_n \left(\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} F^{B;\,n}_{A;\,l} \varphi_B \right), \tag{2.8.7}$$

where we have introduced the notation ∇_n for the covariant derivative with respect to the metric of the effective Riemannian space-time. We will get another important identity, if we take into account the fact that the metric tensor of the effective Riemannian space-time is actually constructed out of φ_{ik} and the metric γ_{ik}. The variation of the action function for the matter becomes

$$\delta S_{\mathrm{M}} = \frac{1}{c} \int d^4x \left(\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_{in}} \delta_{\mathrm{L}} \varphi_{in} + \frac{\delta \mathfrak{L}}{\delta \varphi_A} \delta_{\mathrm{L}} \varphi_A \right.$$

$$\left. - \frac{1}{2} t^{in}_{\mathrm{M}} \delta_{\mathrm{L}} \gamma_{in} + \mathrm{Div} \right) = 0, \tag{2.8.8}$$

where

$$t_{\mathrm{M}}^{in} = -2\frac{\Delta\mathfrak{L}_{\mathrm{M}}}{\Delta\gamma_{in}} = -2\left(\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\gamma_{in}} - \gamma^{ip}\gamma^{ns}\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\gamma^{ps}}\right)$$

is the symmetric tensor of the energy-momentum of matter in a pseudo-Euclidean space-time.

The Lie variations of the gravitational field φ_{in} and of the metric are

$$\delta_{\mathrm{L}}\varphi_{in} = -\varphi_{il}D_n\xi^l - \varphi_{nt}D_i\xi^l - \xi^l D_l\varphi_{in},$$

$$\delta_{\mathrm{L}}\gamma_{in} = -\gamma_{il}D_n\xi^l - \gamma_{nl}D_i\xi^l. \tag{2.8.9}$$

After substitution of (2.8.6) and (2.8.9) into (2.8.8), we obtain

$$\delta S_{\mathrm{M}} = \frac{1}{2}\int d^4x\left(\xi^l\left[2D_n\left(\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_{nm}}\varphi_{ml}\right) - D_n t_{\mathrm{M}l}^{n}\right.\right.$$

$$\left.\left. - \frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_{nm}}D_l\varphi_{nm} - D_n\left(\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_A}F_{A;l}^{B;n}\varphi_B\right) - \frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_A}D_l\varphi_A\right] + \mathrm{Div}\right) = 0$$

Because the vector ξ^l and its derivatives are arbitrary, we will get another strong conservation law

$$D_n t_{\mathrm{M}l}^{n} - 2D_n\left(\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_{nm}}\varphi_{ml}\right) + \frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_{nm}}D_l\varphi_{nm}$$

$$+ D_n\left(\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_A}F_{A;l}^{B;n}\varphi_B\right) + \frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_A}D_l\varphi_A = 0. \tag{2.8.10}$$

Subtracting from this the expression (2.8.7), we get

$$D_n t_{\mathrm{M}l}^{n} - 2D_n\left(\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_{nm}}\varphi_{ml}\right) + \frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_{nm}}D_l\varphi_{nm} = g_{il}\,\Delta_n\,T^{in}. \tag{2.8.11}$$

This identity is valid even if the equation of motion of matter in a gravitational field is not obeyed, and so it is a strong conservation law.

Similarly, since the action function of a gravitational field is invariant under transformation (2.8.3), we will obtain

$$D_n t_{gl}^{n} - 2D_n\left(\frac{\delta\mathfrak{L}_g}{\delta\varphi_{nm}}\varphi_{ml}\right) + \frac{\delta\mathfrak{L}_g}{\delta\varphi_{nm}}D_l\varphi_{mn} = 0, \tag{2.8.12}$$

where the density of the energy-momentum tensor of the gravitational

field in the pseudo-Euclidean space-time, as usual, will be

$$t_g^{in} = -2\frac{\Delta \mathfrak{L}_g}{\Delta \gamma_{in}} = -2\left[\frac{\delta \mathfrak{L}_g}{\delta \gamma_{in}} - \gamma^{ip}\gamma^{ns}\frac{\delta \mathfrak{L}_g}{\delta \gamma^{sp}}\right].$$

Here $t^{n}_{g\,l} = \gamma_{il} t_g^{in}$.

It follows from (2.8.11) and (2.8.12) that

$$D_n[t^n_{gl} + t^n_{Ml}] - 2D_n\left[\frac{\delta \mathfrak{L}}{\delta \varphi_{nm}}\varphi_{ml}\right] + \frac{\delta \mathfrak{L}}{\delta \varphi_{mn}} D_l \varphi_{mn}$$

$$= \nabla_n (g_{il} T^{in}). \tag{2.8.13}$$

When the equations of the gravitational field

$$\frac{\delta \mathfrak{L}}{\delta \varphi_{mn}} = \frac{\delta \mathfrak{L}_g}{\delta \varphi_{mn}} + \frac{\delta \mathfrak{L}_M}{\delta \varphi_{mn}} = 0, \tag{2.8.14}$$

where

$$\frac{\delta \mathfrak{L}}{\delta \varphi_{mn}} = \frac{\partial \mathfrak{L}}{\partial \varphi_{mn}} - \partial_p\left(\frac{\partial \mathfrak{L}}{\partial(\partial_p \varphi_{mn})}\right) + \partial_p \partial_q\left(\frac{\partial \mathfrak{L}}{\partial(\partial_p \partial_q \varphi_{mn})}\right)$$

$$- \partial_p \partial_q \partial_l\left(\frac{\partial \mathfrak{L}}{\partial(\partial_p \partial_q \partial_e \varphi_{mn})}\right), \tag{2.8.15}$$

are satisfied, the expression (2.8.13) becomes simpler:

$$D_n[t^n_{gl} + t^n_{Ml}] = g_{il} \nabla_n T^{in}. \tag{2.8.16}$$

This relation is a manifestation of the principle of geometrization. It suggests that the covariant divergence in a pseudo-Euclidean time-space of the total density of energy-momentum tensor for gravitational field and matter has transformed into the covariant divergence in a Riemannian space-time for the density of energy-momentum tensor of matter defined in the effective Riemannian space. With the field approach the gravitational field (as a physical field) can thus be excluded from describing the motion of matter and its energy, so to speak, goes into forming the effective Riemannian space-time.

The Riemannian space-time will thus be a kind of energy-momentum carrier. According to the principle of geometrization and the laws of energy-momentum conservation for matter and gravitational field taken together, the creation of a Riemannian space-time takes up as much energy as it is contained in the gravitational field, and therefore the propagation of curvature waves in a Riemannian space-time reflects the conventional transfer of energy by gravitational waves in a pseudo-Euclidean space-time. This implies that with the field approach curva-

ture waves in the effective Riemannian space-time are a direct consequence of the existence of Faraday – Maxwell gravitational waves with their energy-momentum density.

When the equations of motion of matter

$$\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} = 0 \tag{2.8.17}$$

hold, the expression (2.8.10) becomes simpler

$$D_n t^n_{\mathrm{M}t} - 2D_n \left(\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_{mn}} \varphi_{ml} \right) + \frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_{mn}} D_l \varphi_{mn} = 0, \tag{2.8.18}$$

and from (2.8.7) we automatically obtain the covariant equation of conservation for the density of the energy-momentum tensor for matter in a Riemannian space-time

$$\Delta_n T^{in} = \partial_n T^{in} + \Gamma^i_{mn} T^{mn} = 0. \tag{2.8.19}$$

This equation is common for theories with geometrical density of the matter Lagrangian and it is not connected with any specific version of gravity theory.

We will see later in the book that, if the equations of gravitational field (2.8.14) hold, we will obtain from (2.8.13) and (2.8.19) the covariant conservation law for densities of the total symmetrical energy-momentum tensor in a pseudo-Euclidean space-time

$$D_n (t^n_{gl} + t^n_{\mathrm{M}l}) = 0. \tag{2.8.20}$$

Expression (2.8.20) taken together with the properties of homogeneity and isotropicity of a pseudo-Euclidean space-time, enables us to construct for a closed system all the integral conserved quantities. This means that this scheme does not include any processes (whatever the erudition of their inventor) in which energy-momentum is not conserved.

It follows also from the expression (2.8.20) that the gravitational field considered in a pseudo-Euclidean space-time behaves just like all the other physical fields. It has an energy and a momentum, and makes its contribution to the density of the total energy-momentum tensor of the system.

From (2.8.20) and (2.8.16) we will get

$$D_n (t^n_{gl} + t^n_{\mathrm{M}l}) = g_{il} \Delta_n T^{in} = 0. \tag{2.8.21}$$

Consequently, the conservation law for the density of the total energy-momentum tensor (2.8.20) and the conservation law in the form (2.8.19), when the equations of motion for matter (2.8.17) and gravitational field (2.8.14) are met, are merely different forms of the same

conservation law. The conservation law (2.8.20) expresses the fact that in a pseudo-Euclidean space-time is conserved the density of the total energy-momentum tensor for matter and gravitational field taken together. This law has the conventional form of a conservation law. A conservation law in a Riemannian space-time is no conservation law in the conventional sense, since the density of the energy-momentum tensor of matter T^{in} need not be conserved, because $\partial_n T^{in} \neq 0$. In that case, the second term in (2.8.19) expresses the energy-related action of the gravitational field on matter and shows that matter derives energy that has been "stored", as it were, in the effective Riemannian space. But there is no saying from (2.8.19) which quantity is con served then, and it is only the equality (2.8.16) that enables us to find that the conserved quantity is the total energy-momentum tensor of the system. We have not yet elaborated on the choice of the Lagrangian density. Within the framework of the special theory of relativity we have used the notion of gravitational field as a Faraday – Maxwell field that possesses energy, momentum and spins 2 and 0 to construct [28-30], using the principle of geometrization, a unique relativistic theory of gravitation (RTG). This theory differs fundamentally from the general theory of relativity of Einstein, since it allows a description of the entire body of experimental gravitational evidence available to date satisfies the principle of conformity and is not fraught with the difficulties that are inherent in the general theory of relativity in relation to the problem of energy-momentum. According to this theory the Friedmannian universe is infinite and "plane". It follows that the Universe must include some "latent mass" in some form of matter. According to RTG the process of compression of a massive body by gravitational forces terminates at some finite density of matter in a finite span of proper time. The brightness of such an object declines exponentially, but nothing special occurs with the object, because its density is always finite and, say, for the body's mass of 10^8 solar masses it is 2 g/cm^3. This means that RTG does not predict any "black holes" as objects whose density grows indefinitely so that they shrink to a point.

Conservation laws do not work in the entire subclass of gravity theories with complete geometrization, not only in Einstein's theory. In theories with complete geometrization the Lagrangian density depends on the field φ_{ik} and the metric tensor γ_{ik} only through the metric tensor of a Riemannian space-time

$$\mathfrak{L} = \mathfrak{L}_g(g_{ik}) + \mathfrak{L}_M(g_{ik}, \varphi_A). \tag{2.8.22}$$

In theories of this subclass the density of the symmetric energy-mo-

mentum tensor for matter and gravitational field in a pseudo-Euclidean space-time is

$$-\frac{1}{2}t^{in}=\frac{\Delta\mathfrak{L}}{\Delta\gamma_{in}}=\frac{\Delta\mathfrak{L}}{\Delta g_{lm}}\,\frac{\partial g_{ln}}{\partial\gamma_{in}}$$

$$-\partial_p\left[\frac{\Delta\mathfrak{L}}{\Delta g_{lm}}\,\frac{\partial g_{lm}}{\partial(\partial_p\gamma_{in})}-\partial_q\left(\frac{\Delta\mathfrak{L}}{\Delta g_{lm}}\,\frac{\partial g_{lm}}{\partial(\partial_p\gamma_{in})}\right)\right]$$

$$-\gamma^{is}\gamma^{np}\left(\frac{\Delta\mathfrak{L}}{\Delta g_{lm}}\,\frac{\partial g_{lm}}{\partial\gamma^{sp}}-\partial_q\left[\frac{\Delta\mathfrak{L}}{\Delta g_{lm}}\,\frac{\partial g_{lm}}{\partial(\partial_q\gamma^{sp})}\right.\right.$$

$$\left.\left.-\partial_k\left(\frac{\Delta\mathfrak{L}}{\Delta g_{lm}}\,\frac{\partial g_{lm}}{\partial(\partial_{kq}\gamma^{sp})}\right)\right]\right). \tag{2.8.23}$$

In any gravity theory with the Lagrangian density (2.8.22) the equations of gravitational field have the form

$$\frac{\Delta\mathfrak{L}}{\Delta g_{lm}}=\frac{\delta\mathfrak{L}}{\delta g_{lm}}-g^{lp}g^{ms}\,\frac{\delta\mathfrak{L}}{\delta g^{sp}}=0, \tag{2.8.24}$$

and so the density of the symmetric tensor of energy-momentum of matter and gravitational field in a pseudo-Euclidean space-time (2.8.23), by virtue of the gravitational field equations (2.8.24), will vanish

$$\frac{\Delta\mathfrak{L}}{\Delta\gamma_{in}}=-\frac{1}{2}t^{in}=0.$$

Therefore, the subclass of gravity theories with the Lagrangian (2.8.22) in principle does not allow to introduce the concept of the gravitational field with an energy and a momentum. This proposition has the nature of a theorem whose consequence is the conclusion that any way of constructing a gravitation theory on the basis of a plane space-time with complete geometrization that leans on the ideas of a gravitational field as a physical field in principle cannot lead to the general theory of relativity. This general conclusion goes to prove the fallacy of some statements [32-33] that all such theories are bound to lead to Einstein's general theory of relativity. Detailed analysis [31, 37, 44] shows that in the general theory of relativity there are no matter conservation laws for a closed system. Thus, for the conservation laws to hold good, we must forgo the complete geometrization of the gravitational field, whose equations can even be nonlinear for the higher derivatives as well.

Then, in full compliance with geometrization principle the field φ^{ik} and the metric tensor of the Minkowskian space γ^{ik} will only enter

into the density of the matter Lagrangian through the quantity g^{ik}, and in the density of the Lagrangian of the gravitational field they will appear separately.

In GTR the gravitational field is characterized by the tensor g^{ik}. It is on this tensor that the boundary conditions at infinity are imposed in disregard of the fact that the asymptotic behavior of metric coefficients is conditioned by the choice of a three-dimensional (space) coordinate system in this reference frame. The latter of necessity suggests that the quantities g^{ik} cannot be characteristics of a Faraday–Maxwell gravitational field, which has an energy and a momentum. Therefore, when we construct a relativistic theory of gravitation, we should reject Einstein's idea that a gravitational field should be identified with the metric of a Riemannian space-time.

Chapter 3

RELATIVISTIC THEORY OF GRAVITATION

This chapter is concerned with the essentials of a relativistic theory of gravitation (RTG) constructed in [28-30]. Before laying down the basics of RTG, we will give an outline of some points of principle associated with the general theory of relativity (GTR).

When expounding the general theory of relativity Einstein relied on the principle of equivalence of the forces of inertia and gravity. The equivalence principle for the forces he formulated as follows [8]: "For an infinitesimal region we can always chose coordinates so that there will be no gravitational field in it." Einstein's formulation of the equivalence principle already forgoes the notion that a gravitational field is a Faraday–Maxwell field. One manifestation of this is the pseudo-tensor characteristic τ_p^l of the gravitational field he introduced. Later on Schrödinger [35] indicated that with an appropriate choice of the coordinate system all the components of the energy-momentum pseudo-tensor of the gravitational field, τ_p^l beyond a ball vanish. Einstein commented [8]: "As regards Schrödinger's ideas, their power lies in their analogy with electrodynamics, where the density and strengths of the energy of any field are nonzero. However, I do not see the reason why the same should be characteristic of gravitational fields as well. Gravitational fields can be defined without specifying strengths and density of energy." We thus see that Einstein quite consciously departed from the concept of the gravitational field as a Faraday–Maxwell field, since this field, as a material entity, can never be done away with by a choice of a reference system.

Since GTR does not include the concept of the density of energy-momentum tensor for the gravitational field, we cannot introduce in it the law of energy-momentum conservation for matter and gravitational field taken together. This was first noted by Hilbert. He wrote [36]: "I maintain . . . that for the general theory of relativity, i.e., when the Hamilton function is generally invariable, there are no

energy equations in orthogonally invariant theories. I could even note this circumstance as a characteristic feature of the general theory of relativity." Some authors do not grasp this even now, others understand and regard this as a step of principle and importance, which has even been made by GTR, by having dethroned such concepts as energy. The rejection of the concept of the density of energy-momentum of the gravitational field in GTR makes it impossible to localize the energy of the field. But if we cannot localize the energy of the field and conservation laws, we cannot use the concepts of gravitational waves and the flux of gravitational radiation. This means that any transfer of gravitational energy in space from one body to another is impossible.

According to the ideology of GTR the principle of relativity is not applicable to gravitational phenomena. It was in this key point that nearly 70 years ago Einstein and Hilbert in their GTR reasoning departed in principle from the special theory of relativity, which in turn means a demise of the conservation laws for energy-momentum and angular momentum. This also predetermined the emergence of the nonphysical ideas that gravitational energy is nonlocalizable and other ideas that have nothing to do with gravity. These two great scientists left the beautifully simple Minkowskian space, which exhibit the maximum (ten-parameter) group of motion of space and ventured instead into the tangle of Riemannian geometry, and so generations of gravitational physicists have been floundering there to date.

If thus one adopts GTR, one must forsake both the fundamental principle – the conservation of energy-momentum of matter and gravitational field – and the concepts of the classical field. This sacrifice is too large, however. It would be too cavalier a move to accept this without adequately checking things experimentally. The only outcome is thus to do without GTR.

It has been shown [31, 37-41] that, since GTR does not (and cannot) provide conservation of energy-momentum for matter and gravitational field together, the inertial mass defined in Einstein's theory has no physical meaning, and the flux of gravitational radiation, as defined in GTR, can always be cancelled just by adequately choosing a permissible reference frame, and hence the quadruple form due to Einstein for the gravitational field is no consequence of GTR. In principle, the general theory of relativity does not suggest that a double system loses energy through gravitational radiation. GTR has no classical Newtonian limit, and so it does not satisfy one of the most fundamental principles of physics – the principle of conformity. These are some inferences from the fact that GTR has no energy-momentum conservation made if one frees oneself of dogmatism, gives a serious

thought to the problem and carries out a detailed physical analysis.

This all indicates that General Relativity is an unsatisfactory physical theory; therefore we will urgently need a classical theory of gravitation such that would meet all the requirements to a physical theory.

Unlike GTR, our theory resides on the principle of relativity that has been put forward by Poincaré as a universal principle for all physical processes. He formulated it as follows [42]: "The laws of physical phenomena will be the same both for an observer at rest and for an observer in uniform translational motion, so that we have no, and even cannot have, means to say whether we are in such a motion or not."

It might appear that, formulated in this manner, the relativity principle cannot be applied to accelerated reference frames. Moreover, Einstein held that one then has to go over to GTR. This is not so, however. It was found [43] that the discovery by Minkowski of the pseudo-Euclidean geometry of space-time makes it possible to formulate a unified principle of relativity: "Whatever physical reference frame (inertial or noninertial) we chose, we can always have an infinite set of other reference frames, such that in them all the physical phenomena occur in the same manner as in the original reference frame, so that we do not (and cannot) have any experimental possibilities to tell to which specific frame out of the infinite set of frames we belong." This implies that when describing physical phenomena in Minkowskian space we can, depending on the physical problem at hand, take any suitable reference frame, adequate to the problem, and hence to define the corresponding metric tensor γ_{lp} in the Minkowskian space. Why didn't Einstein understand this? This seems to be explained by the fact that he perceived Relativity only through the postulate of the constancy of the speed of light in Galilean coordinates, and identified accelerated reference frame with gravitation by the principle of equivalence.

Underlying our theory is the notion of the gravitational field as a physical field *à la* Faraday–Maxwell, one that possesses energy-momentum. A gravitational field is thus similar to all other physical fields in that it is characterized by its energy-momentum tensor of the system. We view the gravitational field as a physical field with spin 2 and 0, and an asymptotically free gravitational field has spin 2. For all physical fields space-time has a pseudo-Euclidean geometry (Minkowskian space).

To sum up: the laws of conservation of energy-momentum and angular momentum strictly hold for a closed system. This is another radical departure of our theory from Einstein's GTR.

A further important question that emerges in connection of gravitation theories is that of interaction of a gravitational field with matter. A gravitational field, according to present thinking, is a universal field:

It acts on all forms of matter in the same manner. We will lay at the basis of our theory the principle of geometrization [43, 44], according to which we can represent identically the equations of motion of matter in a tensor gravitational field φ^{ik} in a Minkowskian space with a metric tensor γ^{ik} the equations of motion of matter in the effective Riemannian space-time with the metric tensor g^{ik}, which depends on the gravitational field φ^{ik} and the metric tensor γ^{ik}. We thereby introduce the concept of the effective Riemannian space of field nature. This force space is created in RTG in strict accordance with the conservation laws; it is due to the presence of the gravitational field and definite, universal character of its action on matter. The curvature of this dynamic Riemannian space emerges, as a secondary concept, owing to the principle of geometrization and is itself a consequence of the action of the gravitational field.

By the principle of geometrization in Minkowski's space we can represent the Lagrangian density as

$$\mathfrak{L} = \mathfrak{L}_g(\tilde{\gamma}^{ik}, \tilde{\varphi}^{ik}) + \mathfrak{L}_M(\tilde{g}^{ik}, \varphi_A) \qquad \text{(A)}$$

where $\tilde{\varphi}^{ik} = \sqrt{-\gamma}\,\varphi^{ik}$ is the density of the tensor of the field variable of the gravitational field φ^{ik}, $\tilde{g}^{ik} = \sqrt{-g}\,g^{ik}$ is the density of the metric tensor of the Riemannian space g^{ik}; $\tilde{\gamma}^{ik} = \sqrt{-\gamma}\,\gamma^{ik}$ is the density of the metric tensor of the Minkowskian space, and φ_A is the matter field.

In this theory the density of the Lagrangian of the gravitational field, $\mathfrak{L}_g$, depends on the metric tensor γ^{ik} and the gravitational field φ^{ik} therefore it is radically different from GTR, in which the density of the Lagrangian is only dependent on the metric tensor of the Riemannian space g^{ik}. The density of the Lagrangian of the gravitational field in our theory is thus not completely geometrized, whereas in GTR is.

We will see later in the book, the concept of the gravitational field that has a density of energy-momentum and spin 2 and 0 as well as the geometrization principle enable us to give a unique treatment of a relativistic theory of gravitation. The theory changes the current views on space-time that have taken shape under the influence of GTR; it leads us out the maze of Riemannian geometry and is in line with the current theories in particle physics. As a consequence of the theory the general principle of relativity of Einstein is devoid of physical meaning and any content [9].

3.1. Inertial Mass in GTR

The equality of the inertial and gravitational masses of a. body was regarded by Einstein as an exact law of nature, one that had to be reflected in his theory. At present it is thought to be proved that in GTR the gravitational mass (or as it is sometimes called, the heavy mass) of a system consisting of matter and gravitational field is equal to its inertial mass. For example, a statement to this effect is to be found in Einstein [8], Talman [48], and Weyl [49]. Later on some other "proofs" of the theorem with slight modifications were given by other authors [14, 32, 47].

The conclusion is erroneous, however. Following works [31, 34] we will show here there error lies.

The heavy mass M of an arbitrary physical system which is at rest as a whole relative to a Galilean coordinate system that is Schwarzschildian at infinity, was defined by Einstein [8] as a quantity that appears as a factor at the term $-2G/(c^2 r)$ in the asymptotic expression ($r \to \infty$) for the component g_{00} of the metric tensor of the Riemannian space-time

$$g_{00} = 1 - \frac{2G_M}{c^2 r}.$$

A slightly different definition of the gravitational mass was given by Talman [48]

$$M = \frac{c^2}{4\pi G} \int R_0^0 \sqrt{-g}\, dV. \tag{3.1.1}$$

It follows immediately from these definitions that the magnitude of the gravitational mass remains unaltered under transformations of three-dimensional coordinates, since both the component R_0^0 of the Ricci tensor and the component g_{00} of the metric tensor in this case is transformed like a scalar.

In the case of a static, spherically symmetrical source these definitions are equivalent. We now show that they are also equivalent for any statical systems. To see that, we will write the component R_0^0 in the form

$$R_0^0 = g^{0i}\left[\frac{\partial}{\partial x^l}\Gamma_{0i}^l - \frac{\partial}{\partial x^0}\Gamma_{pi}^p + \Gamma_0^n\Gamma_{np}^p - \Gamma_{pi}^n\Gamma_{n0}^p\right].$$

After some identical rearrangements, this yields

$$R_0^0 = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}[\sqrt{-g}g^{0n}\Gamma_{0n}^\alpha] - g^{0i}\frac{\partial}{\partial x^0}\Gamma_{ni}^n -$$

$$-\frac{1}{2}\Gamma_{ni}^0\frac{\partial g^{ni}}{\partial x^0} + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^0}[\sqrt{-g}g^{0n}\Gamma_{0n}^0]. \tag{3.1.2}$$

Since with static systems we can ignore the last three terms, we have from (3.1.1)

$$M = \frac{c^2}{4\pi G} \oint \sqrt{-g}\, g^{0n} \Gamma^{\alpha}_{0n\alpha} S_\alpha . \tag{3.1.3}$$

Far away from the static system its metric with a predetermined accuracy can be described by the Schwarzschild metric, and so expression (3.1.3) becomes

$$M = -\frac{c^2}{8\pi G} \lim_{r\to\infty} \oint g^{00} \sqrt{-g} \frac{\partial g_{00}}{\partial x^\alpha} dS^\alpha . \tag{3.1.4}$$

Since the integrant in (3.1.1) is a scalar under any transformations of a three-dimensional coordinate system, the magnitude of the gravitational mass M will be independent of the choice of coordinates. In Schwarzschild coordinates we have from (3.1.4)

$$M = \frac{c^2}{2G} \lim_{r\to\infty} \left(r^2 \frac{\partial g_{00}}{\partial r} \right) = \frac{c^2}{2G} \lim_{r\to\infty} \left[r^2 \frac{\partial}{\partial r}\left(1 - \frac{2G}{c^2 r} M \right) \right] .$$

According to Talman, the gravitational mass of any static system is thus a factor at the term $-2G/(c^2 r)$ in the asymptotic expression for the component g_{00} of the metric tensor of a Riemannian space-time. Accordingly, the definitions of the heavy mass given by Einstein and Talman for static system coincide.

In Einstein's GTR the concept of the inertial mass of a physical system is closely linked with the concept of the energy of the system [8]: "... the quantity which we have interpreted as energy also plays the role of inertial mass, in accordance with the special theory of relativity." In the general theory of relativity Einstein suggested that the energy of a system should be calculated in terms of pseudo-tensors of energy-momentum, therefore the inertial mass was calculated by

$$m_i = \frac{1}{c} P^0 = \frac{1}{c^2} \int (-g) \left[T^{00} + \tau^{00} \right] dV = \frac{1}{c^2} \oint h^{00\alpha} dS_\alpha .$$

We now determine using this relation the inertial mass of a spherically symmetrical source of gravitational field and examine its transformational properties under coordinate transformations.

In isotropic Cartesian coordinates the metric of a Riemannian space-time has the form

$$g_{00} = \frac{\left(1 - \dfrac{r_g}{4r}\right)^2}{\left(1 + \dfrac{r_g}{4r}\right)^2} , \quad g_{\alpha\beta} = \gamma_{\alpha\beta} \left(1 + \frac{r_g}{4r} \right)^4 , \tag{3.1.5}$$

where

$$r_g = \frac{2G}{c^2} M.$$

These coordinates are asymptotically Galilean, since as $r \to \infty$ we have the estimates

$$g_{00} = 1 + O\left(\frac{1}{r}\right), \quad g_{\alpha\beta} = \gamma_{\alpha\beta}\left(1 + O\left(\frac{1}{r}\right)\right). \tag{3.1.6}$$

Using the covariant components of the metric (3.1.5), from the expression

$$h^{ikl} = \frac{c^4}{16\pi G}\frac{\partial}{\partial x^m}\left[-g(g^{ik}g^{ml} - g^{il}g^{mk})\right]$$

we have

$$h^{00\alpha} = -\frac{c^4}{16\pi G}\frac{\partial}{\partial x^\beta}\left[g_{11}g_{22}g_{33}g^{\alpha\beta}\right].$$

Substituting this expression into the relation

$$P^i = \frac{1}{c}\oint h^{0i\alpha} dS_\alpha = \text{const}, \tag{3.1.7}$$

taking into account that

$$dS_\alpha = -\frac{x_\alpha}{r} r^2 \sin\Theta d\Theta d\varphi,$$

and integrating over an infinitely remote surface, we will get

$$P^0 = \frac{c^3}{16\pi G}\lim_{r\to\infty} r^2 \int \frac{x_\alpha}{r}\frac{\partial}{\partial x^\beta}\left[-g_{11}g_{22}g_{33}g^{\alpha\beta}\right] \sin\Theta d\Theta d\varphi. \tag{3.1.8}$$

The component P^0 is thus independent of the component g_{00} of the metric tensor of the Riemannian space-time. From (3.1.5) and (3.1.8) and the relation

$$\frac{\partial}{\partial x^\beta} f(r) = -\frac{x_\beta}{r}\frac{\partial}{\partial r} f(r), \tag{3.1.9}$$

where

$$x_\alpha x^\alpha = -r^2,$$

we have

$$P^0 = \frac{c^3 r_g}{2G} = Mc. \tag{3.1.10}$$

It is this coincidence of the "inertial mass" with the heavy mass that gave grounds for maintaining that they are equal in GTR [47]: "... $P^\alpha = 0, P^0 = Mc$ is a result that was to be expected. It is the manifestation of the fact of the equality of, the so-called 'heavy' and 'inertial' masses (the 'heavy' mass is the mass that determines the gravitational field produced by a body; it is the mass that enters into the metric tensor in the gravitational field, or, specifically, into Newton's law; on the other hand, the 'inertial' mass defines the relationship between the momentum and energy of the body and, specifically, the rest energy of the body equals that mass multiplied by c^2)."

But this statement of Einstein [8] and those of other authors [14, 32, 46-48] are not correct. Obviously the "energy" of the system, and hence its "inertial mass", have no physical meaning whatsoever, since their values also depend on the choice of a three-dimensional coordinate system.

Indeed, one elementary requirement to be met by the definition of the inertial mass is the condition that its magnitude should be independent of the choice of a three-dimensional coordinate system, which occurs in any physical theory. In GTR, however, the definition of the "inertial mass" does not meet this requirement.

We now show, for example, that in the case of the Schwarzchild solution the "inertial mass" can assume any values, depending on the choice of a system of space coordinates. To begin with, we change from three-dimensional Cartesian coordinates x^α_{old} to other coordinates x^α_{new} related to the old ones by

$$x^\alpha_{\text{old}} = x^\alpha_{\text{new}}[1 + f(r_{\text{new}})], \tag{3.1.11}$$

where

$$r_{\text{new}} = \sqrt{x^2_{\text{new}} + y^2_{\text{new}} + z^2_{\text{new}}}$$

and $f(r_{\text{new}})$ is an arbitrary nonsingular function satisfying the conditions

$$f(r_{\text{new}}) \geqslant 0, \qquad \lim_{r_{\text{new}} \to \infty} f(r_{\text{new}}) = 0,$$

$$\lim_{r_{\text{new}} \to \infty} r_{\text{new}} \frac{\partial}{\partial r_{\text{new}}} f(r_{\text{new}}) = 0. \tag{3.1.12}$$

We can readily see that the transformation (3.1.11) corresponds to a change in arithmetization of points of the three-dimensional space along the radius

$$r_{\text{old}} = r_{\text{new}}[1 + f(r_{\text{new}})].$$

For the transformation (3.1.11) to have an inverse one and to be a one-

to-one transformation, it is necessary and sufficient that

$$\frac{\partial r_{\text{old}}}{\partial r_{\text{new}}} = 1 + f + r_{\text{new}} f' > 0,$$

where

$$f' = \frac{\partial}{\partial r_{\text{new}}} f(r_{\text{new}}).$$

Then, the Jacobian of the transformation will be nonzero

$$J = \det \left\| \frac{\partial x_{\text{old}}}{\partial x_{\text{new}}} \right\| = (1 + f)^2 \frac{\partial r_{\text{old}}}{\partial r_{\text{new}}} \neq 0.$$

In particular, all the above requirements are met by the function

$$f(r_{\text{new}}) = \alpha^2 \sqrt{\frac{8GM}{c^2 r_{\text{new}}}} \, [1 - \exp(-\epsilon^2 r_{\text{new}})], \tag{3.1.13}$$

where α and ϵ are arbitrary nonzero numbers.

Since in this case

$$\frac{\partial r_{\text{old}}}{\partial r_{\text{new}}} = 1 + \alpha^2 \sqrt{\frac{8GM}{c^2 r_{\text{new}}}} \left[\frac{1}{2} + \left(\epsilon^2 r_{\text{new}} - \frac{1}{2} \right) \exp(-\epsilon^2 r_{\text{new}}) \right]$$

when r_{old} is a monotône function of r_{new}. We can easily see that $f(r_{\text{new}})$ is a nonnegative, nonsingular function throughout the space. The Jacobian will then be strictly larger than unity

$$J = (1 + f)^2 \frac{\partial r_{\text{old}}}{\partial r_{\text{new}}} > 1.$$

Therefore, the transformation (3.1.11) with the function $f(r_{\text{new}})$, given by (3.1.13) has an inverse transformation and is a one-to-one transformation.

Clearly, the gravitational mass (3.1.1) will not change under transformation (3.1.11). We will now calculate the "inertial mass" in the new coordinates x^α_{new}. Using the law of transformation of the metric tensor

$$g_{ni}^{\text{new}} = \frac{\partial x^l_{\text{old}}}{\partial x^n_{\text{new}}} \frac{\partial x^m_{\text{old}}}{\partial x^i_{\text{new}}} g_{ml}^{\text{old}} (x_{\text{old}}(x_{\text{new}})), \tag{3.1.14}$$

we will find the components of the Schwarzschild metric (3.1.5) in new

coordinates. As a result, we will get

$$g_{00} = \left[1 - \frac{r_g}{4r_{\text{new}}(1+f)}\right]^2 \left[1 + \frac{r_g}{4r_{\text{new}}(1+f)}\right]^{-2},$$

$$g_{\alpha\beta} = \left[1 + \frac{r_g}{4r_{\text{new}}(1+f)}\right]^4 \left\{-\delta_{\alpha\beta}(1+f)^2\right.$$

$$\left. - x_\alpha^{\text{new}} x_\beta^{\text{new}} \left[(f')^2 + \frac{2}{r_{\text{new}}} f'(1+f)\right]\right\}. \tag{3.1.15}$$

The determinant of the metric tensor (3.1.15) is

$$g = -g_{00}\left[1 + \frac{r_g}{4r_{\text{new}}(1+f)}\right]^{12} (1+f)^4$$

$$\times \left[(1+f)^2 + r_{\text{new}}^2(f')^2 + 2r_{\text{new}} f'(1+f)\right]. \tag{3.1.16}$$

It should be emphasized that the metric (3.1.15) is an asymptotically Galilean one, i.e.,

$$\lim_{r_{\text{new}} \to \infty} g_{00} = 1, \quad \lim_{r_{\text{new}} \to \infty} g_{\alpha\beta} = \gamma_{\alpha\beta}.$$

In the special case that the function f is defined by the relation (3.1.13) and $r_{\text{new}} \to \infty$, the metric of the Riemannian space-time will have the following asymptotic behavior:

$$g_{00} \simeq 1 + O\left(\frac{1}{r_{\text{new}}}\right), \quad g_{\alpha\beta} = \gamma_{\alpha\beta} + O\left(\frac{1}{\sqrt{r_{\text{new}}}}\right). \tag{3.1.17}$$

The covariant components of the metric (3.1.15) will be

$$g^{00} = \frac{1}{g_{00}}, \quad g^{\alpha\beta} = \gamma^{\alpha\beta} A + x^\alpha_{\text{new}} x^\beta_{\text{new}} B, \tag{3.1.18}$$

where

$$A = (1+f)^{-2}\left[1 + \frac{r_g}{4r_{\text{new}}(1+f)}\right]^{-4},$$

$$B = \frac{r_{\text{new}}(f')^2 + 2f'(1+f)}{r_{\text{new}}(1+f)^2\left[(1+f)^2 + r_{\text{new}}^2(f')^2 + 2r_{\text{new}} f'(1+f)\right]}$$

$$\times \left[1 + \frac{r_g}{4r_{\text{new}}(1+f)}\right]^{-4}$$

Substituting expressions (3.1.16) and (3.1.18) into (3.1.8) gives

$$P^0 = \frac{c}{16\pi G} \lim_{r_{\text{new}} \to \infty} r_{\text{new}}^2 \int \frac{x_\alpha^{\text{new}}\, \partial}{r_{\text{new}} \partial x_{\text{new}}^\beta}$$
$$\times \left\{ -\delta^{\alpha\beta}(1+f)^2 \left[1 + \frac{r_g}{4r_{\text{new}}(1+f)} \right]^8 \right.$$
$$\times [(1+f)^2 + r_{\text{new}}^2 (f')^2 + 2r_{\text{new}} f'(1+f)] + \frac{x_{\text{new}}^\alpha x_{\text{new}}^\beta}{r_{\text{new}}^2} (1+f)^2$$
$$\left. \times \left[1 + \frac{r_g}{4r_{\text{new}}(1+f)} \right]^8 [r_{\text{new}}^2 (f')^2 + 2r_{\text{new}} f'(1+f)] \right\} dV.$$

By (3.1.9), we have

$$P^0 = \frac{c^3}{2G} \lim_{r_{\text{new}} \to \infty} \left\{ r_{\text{new}}^3 (f')^2 (1+f)^2 \right.$$
$$\times \left[1 + \frac{r_g}{4r_{\text{new}}(1+f)} \right]^8 + r_g (1+f)^2 (1+f+r_{\text{new}} f')$$
$$\left. \times \left[1 + \frac{r_g}{4r_{\text{new}}(1+f)} \right]^7 \right\}. \tag{3.1.19}$$

Using the asymptotic relation (3.1.12) for f, we arrive at

$$P^0 = \frac{c^3}{2G} \lim_{r_{\text{new}} \to \infty} \{ r_g + r_{\text{new}}^3 (f')^2 \}. \tag{3.1.20}$$

The value of the "inertial mass" thus substantially depends on how fast f' tends to zero as $r_{\text{new}} \to \infty$. In particular, we can take $f(r_{\text{new}})$ in the form (3.1.13) and then we will have from (3.1.20)

$$m = M(1 + \alpha^4). \tag{3.1.21}$$

It follows that for the "inertial mass" of the system consisting of matter and gravitational field we in GTR can obtain, since α is arbitrary, any predetermined number $m \geqslant M$ depending on our choice of space coordinates, although the gravitational mass (3.1.1) M of the system, and hence all the three effects of GTR, will remain unchanged in the process. We note also that under more complicated transformations of space coordinates under which the metric remains Galilean, the "inertial mass" of the system can assume any predetermined values, both positive and negative.

We see, thus, that in GTR value of the "inertial mass", which was first introduced by Einstein and adopted then by other authors [14, 32,

46-48], is dependent on the choice of the three-dimensional coordinate system, and so it is devoid of any physical significance. Consequently, the statement that the "inertial" and "heavy" masses are equal in Einstein's theory has no physical meaning as well. This equality holds for a narrow class of three-dimensional coordinate systems, and since the "inertial" and gravitational (3.1.1) masses are governed by different transformational laws, in passing to other three-dimensional coordinate systems their equality no longer holds.

Furthermore, this definition of the "inertial mass" in GTR does not satisfy the principle of conformity with Newton's theory. Indeed, because the "inertial mass" m in Einstein's theory depends on the choice of a three-dimensional coordinate system, its expression in the general case of an arbitrary three-dimensional coordinate system will not go into a corresponding expression in Newton's theory, in which the "inertial mass" is independent of the choice of space coordinates. In consequence, in GTR there is no classical Newtonian limit, and so it does not meet the principle of conformity.

In this connection, a question presents itself: how come that to date the absurdity of the definition

$$P^i = \frac{1}{c} \oint h^{0i\alpha} dS_\alpha = \text{const}$$

of the "energy-momentum" of a system and its "inertial mass" in GTR have not been revealed?

This can only be accounted for by the fact that generally all computations of the "energy-momentum" and "inertial mass" were performed in some narrow class of three-dimensional coordinate systems where "inertial" and gravitational masses do coincide.

In the same class of coordinate systems the expression for the "inertial mass" in Newtonian approximation coincides with the corresponding expression in Newton's theory, which gave rise to the illusion that there exists a classical limit in GTR. And nobody cared to give some thought to the physical meaning of the "inertial mass" introduced in GTR.

3.2. Geometrization and General Relations in RTG

Without loss of generality, we will believe that the tensor density of the metric tensor of the Riemannian space-time $\tilde{g}^{ik}$ is a local function, which only depends on the density of the metric tensor of the Minkowskian space $\tilde{\gamma}^{ik}$ and the density of the tensor of the gravitational field $\tilde{\varphi}^{ik}$.

We will assume that the density of the matter Lagrangian $\mathfrak{L}_M$ is only dependent on the fields φ_A, their covariant first-order derivatives,

and also, by the principle of geometrization, on the density of the metric tensor $\tilde{g}^{ik}$. We will further assume that the density of the Lagrangian of the gravitational field is dependent on the density of the metric tensor $\tilde{\gamma}^{ik}$, its first-order derivatives, and also on the density of the gravitational field $\tilde{\varphi}^{ik}$ and its first-order covariant derivatives with respect to Minkowski's metric. To derive the conservation laws we will make use of the invariance property of the action under infinitesimal displacements of coordinates. Since for any specified Lagrangian density $\mathfrak{L}$ the action

$$S = \int \mathfrak{L} d^4x$$

is a scalar, for an arbitrary infinitesimal coordinate transformation the variation δS will be zero.

To begin with, we calculate the variation of the action of matter

$$S_{\mathrm{M}} = \int \mathfrak{L}_{\mathrm{M}} d^4x$$

under the transformation

$$x'^i = x^i + \xi^i(x), \tag{3.2.1}$$

where $\xi^i(x)$ is an infinitesimal four-vector of displacement,

$$\delta S_{\mathrm{M}} = \int d^4x \left[\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \tilde{g}^{mn}} \delta_L \tilde{g}^{mn} + \frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} \delta_L \varphi_A + \mathrm{Div} \right] = 0. \tag{3.2.2}$$

In (3.2.) Div stands for the divergent terms, which in this section are of no consequence for our reasoning.

The Euler variation is defined, as usual, by

$$\frac{\delta \mathfrak{L}}{\delta \varphi} \equiv \frac{\partial \mathfrak{L}}{\partial \varphi} - \partial_n \frac{\partial \mathfrak{L}}{\partial(\partial_n \varphi)} + \partial_n \partial_k \frac{\partial \mathfrak{L}}{\partial(\partial_n \partial_k \varphi)} - \ldots$$

The variations $\delta_{\mathrm{L}} \tilde{g}^{mn}$ and $\delta_{\mathrm{L}} \varphi_A$ under coordinate transformations (3.2.1) can be readily found, if we use the appropriate transformation law:

$$\delta_{\mathrm{L}} \tilde{g}^{mn} = \tilde{g}^{kn} D_k \xi^m + \tilde{g}^{km} D_k \xi^n - D_k(\xi^k \tilde{g}^{mn}), \tag{3.2.3}$$

$$\delta_{\mathrm{L}} \varphi_A = -\xi^k D_k \varphi_A + F^{B;n}_{A;k} \varphi_B D_n \xi^k. \tag{3.2.4}$$

Hereinafter D_k is the covariant derivative with respect to the Minkowskian metric. Substituting these expressions into (3.2.2) and integrating

by parts, we will get

$$\delta S_{\mathrm{M}} = \int d^4x \left\{ -\xi^m \left[D_k\left(2\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \tilde{g}^{mn}} \tilde{g}^{kn} \right) - D_m \left(\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \tilde{g}^{lp}} \right) \tilde{g}^{lp} + D_k \left(\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} F^{B;k}_{A;m} \varphi_B \right) + \frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} D_m \varphi_A \right] + \mathrm{Div} \right\} = 0.$$

The vector ξ^m being arbitrary, we obtain from the condition $\delta S_{\mathrm{M}} = 0$ the strong identity

$$D_k\left(2\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \tilde{g}^{mn}} \tilde{g}^{kn} \right) - D_m \left(\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \tilde{g}^{lp}} \right) \tilde{g}^{lp} = - D_k \left(\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} F^{B;k}_{A;m} \varphi_B \right) - \frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \varphi_A} D_m \varphi_A, \tag{3.2.5}$$

which holds even if the equations of motion for the fields are not obeyed.

We define

$$T_{mn} = 2\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta g^{mn}}, \quad T^{mn} = -2\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta g_{mn}} = g^{mk} g^{np} T_{kp}, \tag{3.2.6a}$$

$$\tilde{T}_{mn} = 2\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \tilde{g}^{mn}}, \quad \tilde{T}^{mn} = -2\frac{\delta \mathfrak{L}_{\mathrm{M}}}{\delta \tilde{g}_{mn}} = \tilde{g}^{mk} \tilde{g}^{np} \tilde{T}_{kp}. \tag{3.2.6b}$$

Here T_{mn} is the density of the energy-momentum tensor for matter in a Riemannian space and is called the Hilbert tensor density.

We can, by (3.2.6b), represent the left-hand side of (3.2.5)

$$D_k(\tilde{T}_{mn}\tilde{g}^{kn}) - \frac{1}{2}\tilde{g}^{kp} D_m \tilde{T}_{kp} = \partial_k(\tilde{T}_{mn}\tilde{g}^{kn}) - \frac{1}{2}\tilde{g}^{kp}\partial_m \tilde{T}_{kp}.$$

The right-hand side of this can readily be reduced to

$$\partial_k(\tilde{T}_{mn}\tilde{g}^{kn}) - \frac{1}{2}\tilde{g}^{kp}\partial_m \tilde{T}_{kp} = \tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn} - \frac{1}{2}\tilde{g}^{kn}\tilde{T} \right), \tag{3.2.7}$$

where $\tilde{T}=\tilde{g}_{kp}\tilde{T}^{kp}$, ∇_k is the covariant derivative with respect to the metric of a Riemannian space.

The strong identity can, from (3.2.7), be written as

$$\tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn}-\frac{1}{2}\tilde{g}^{kn}\tilde{T}\right)=-D_k\left(\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_A}F_{A;\,m}^{B;\,k}\varphi_B\right)$$

$$-\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_A}D_m\varphi_A. \tag{3.2.8}$$

By the least-action principle, the equations of motion for matter fields have the form

$$\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_A}=0. \tag{3.2.9}$$

From these equations and (3.2.8) we will find the weak identity

$$\nabla_m\left(\tilde{T}^{mn}-\frac{1}{2}\tilde{g}^{mn}\tilde{T}\right)=0. \tag{3.2.10}$$

Notice that the density of the energy-momentum tensor for matter in a Riemannian space T^{mn} is related to $\tilde{T}^{mn}$ by

$$\sqrt{-g}\,T^{mn}=\tilde{T}^{mn}-\frac{1}{2}\tilde{g}^{mn}\tilde{T}. \tag{3.2.11}$$

Therefore, from (3.2.10) we obtain the covariant equation of matter conservation in a Riemannian space

$$\nabla_m T^{mn}=0. \tag{3.2.12}$$

If there are four equations for the matter field, then and only then instead of the equations for this field (3.2.9) we can always apply the equivalent equations (3.2.12). The variation of the action untegral (3.2.2) can be written in equivalent form

$$\delta S_{\mathrm{M}}=\int d^4x\left\{\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\tilde{\varphi}^{mn}}\delta_L\tilde{\varphi}^{mn}+\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\tilde{\gamma}^{mn}}\delta_L\tilde{\gamma}^{mn}\right.$$

$$\left.+\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\varphi_A}\delta_L\varphi_A+\mathrm{Div}\right\}=0. \tag{3.2.13}$$

The variations $\delta_L\tilde{\varphi}^{mn}$ and' $\delta_L\tilde{\gamma}^{mn}$ under the coordinate transformations (3.2.1) will then be

$$\delta_L\tilde{\varphi}^{mn}=\tilde{\varphi}^{kn}D_k\xi^m+\tilde{\varphi}^{km}D_k\xi^n-D_k(\xi^k\tilde{\varphi}^{mn}), \tag{3.2.14}$$

$$\delta_L\tilde{\gamma}^{mn}=\tilde{\gamma}^{kn}D_k\xi^m+\tilde{\gamma}^{km}D_k\xi^n-\tilde{\gamma}^{mn}D_k\xi^k. \tag{3.2.15}$$

Substituting the expressions for the variations $\delta_L \tilde{\varphi}^{mn}$, $\delta_L \tilde{\gamma}^{mn}$, and $\delta_L \varphi_A$ into (3.2.13) and integrating by parts we will obtain, since ξ^m is arbitrary, the strong identity

$$D_k\left(2\frac{\delta \mathfrak{L}_M}{\delta \tilde{\varphi}^{mn}}\tilde{\varphi}^{kn}\right) - D_m\left(\frac{\delta \mathfrak{L}_M}{\delta \tilde{\varphi}^{kp}}\right)\tilde{\varphi}^{kp} + D_k\left(2\frac{\delta \mathfrak{L}_M}{\delta \tilde{\gamma}^{mn}}\tilde{\gamma}^{kn}\right)$$

$$-D_m\left(\frac{\delta \mathfrak{L}_M}{\delta \tilde{\gamma}^{kp}}\tilde{\gamma}^{kp}\right) = -D_k\left(\frac{\delta \mathfrak{L}_M}{\delta \varphi_A}F^{B;k}_{A;m}\varphi_B\right) - \frac{\delta \mathfrak{L}_M}{\delta \varphi_A}D_m\varphi_A, \tag{3.2.16}$$

which, just like (3.2.15), is valid even if the equations of motion for matter and gravitational field are not obeyed.

For any Lagrangian we will introduce some notation and relations to be used later in our reasoning

$$\tilde{t}^{mn} = -2\frac{\delta \mathfrak{L}}{\delta \tilde{\gamma}_{mn}}, \qquad t^{mn} = -2\frac{\delta \mathfrak{L}}{\delta \gamma_{mn}}, \tag{3.2.17a}$$

$$t^{mn} = \frac{1}{\sqrt{-\gamma}}\left(\tilde{t}^{mn} - \frac{1}{2}\tilde{\gamma}^{mn}\tilde{t}\right). \tag{3.2.17b}$$

Since $\mathfrak{L}_M$, by the principle of geometrization, is only dependent on $\tilde{\gamma}^{mn}$ via $\tilde{g}^{mn}$, we can easily find the relation between $\tilde{t}_{Mmn}$ and $\tilde{T}_{mn}$

$$t_{Mmn} = 2\frac{\delta \mathfrak{L}_M}{\delta \tilde{\gamma}^{mn}} = \tilde{T}_{kp}\frac{\partial \tilde{g}^{kp}}{\partial \tilde{\gamma}^{mn}}. \tag{3.2.18a}$$

We have here taken into account the definition (3.2.6b).

From the identity

$$\frac{\partial \tilde{g}^{kp}}{\partial \tilde{\gamma}_{mn}} = \tilde{\gamma}^{ml}\tilde{\gamma}^{nq}\frac{\partial \tilde{g}^{pk}}{\partial \gamma^{lq}},$$

and (3.2.17a), we find

$$\tilde{t}^{mn}_M = -\tilde{T}_{pk}\frac{\partial \tilde{g}^{pk}}{\partial \tilde{\gamma}_{mn}}. \tag{3.2.18b}$$

If then we take into consideration (3.2.6b) and the relation

$$-\tilde{g}_{lp}\tilde{g}_{qk}\frac{\partial \tilde{g}^{lq}}{\partial \tilde{\gamma}_{mn}} = \frac{\partial \tilde{g}_{pk}}{\partial \tilde{\gamma}_{mn}},$$

we get

$$\tilde{t}^{mn}_M = \tilde{T}^{pk}\frac{\partial \tilde{g}_{pk}}{\partial \tilde{\gamma}_{mn}}. \tag{3.2.18c}$$

Comparing (3.2.8), and (3.2.16), we find from (3.2.17a)

$$\tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn}-\frac{1}{2}\tilde{g}^{kn}\tilde{T}\right)=\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}_{\mathrm{M}}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}_{\mathrm{M}}\right)$$

$$+D_k\left(2\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\tilde{\varphi}^{mn}}\tilde{\varphi}^{kn}\right)-D_m\left(\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\tilde{\varphi}^{kp}}\right)\tilde{\varphi}^{kp}. \qquad (3.2.19)$$

Similarly, because the action of the gravitational field under the coordinate transformation (3.2.1) is invariant, we have

$$\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}_{g}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}_{g}\right)+D_k\left(2\frac{\delta\mathfrak{L}_{g}}{\delta\tilde{\varphi}^{mn}}\tilde{\varphi}^{kn}\right)$$

$$-D_m\left(\frac{\delta\mathfrak{L}_{g}}{\delta\tilde{\varphi}^{kp}}\right)\tilde{\varphi}^{kp}=0. \qquad (3.2.20)$$

Combining (3.2.19) and (3.2.20) gives

$$\tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn}-\frac{1}{2}\tilde{g}^{kn}\tilde{T}\right)=\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}\right)$$

$$+D_k\left(2\frac{\delta\mathfrak{L}}{\delta\tilde{\varphi}^{mn}}\tilde{\varphi}^{kn}\right)-D_m\left(\frac{\delta\mathfrak{L}}{\delta\tilde{\varphi}^{kp}}\right)\tilde{\varphi}^{kp}. \qquad (3.2.21)$$

Here

$$\tilde{t}^{kn}=\tilde{t}^{kn}_{g}+t^{kn}_{\mathrm{M}}. \qquad (3.2.22)$$

From the least-action principle the equations for the gravitational field have the form

$$\frac{\delta\mathfrak{L}}{\delta\tilde{\varphi}^{mn}}=\frac{\delta\mathfrak{L}_{\mathrm{g}}}{\delta\tilde{\varphi}^{mn}}+\frac{\delta\mathfrak{L}_{\mathrm{M}}}{\delta\tilde{\varphi}^{mn}}=0. \qquad (3.2.23)$$

From these equations and (3.2.21) we obtain the following important relation:

$$\tilde{g}_{mn}\nabla_k\left(\tilde{T}^{kn}-\frac{1}{2}\tilde{g}^{kn}T\right)=\tilde{\gamma}_{mn}D_k\left(\tilde{t}^{kn}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}\right). \qquad (3.2.24)$$

Since the density of the total tensor of energy-momentum in Minkowski's space is given by

$$\sqrt{-\gamma}\,t^{kn}=\tilde{t}^{kn}-\frac{1}{2}\tilde{\gamma}^{kn}\tilde{t}, \qquad (3.2.25)$$

we can from this and also from (3.2.11) rewrite (34.24) in the form

$$D_m t_n^m = \nabla_m T_n^m . \tag{3.2.26}$$

This relation reflects the geometrization principle: the covatiant divergence in a pseudo-Euclidean space of the total density of the energy-momentum tensors for matter and gravitational field is exactly equal to the covariant divergence in the effective Riemannian space only for the density of the energy-momentum tensor for matter. When the equations of motion for matter hold, we have

$$D_m t_n^m = \nabla_m T_n^m = 0. \tag{3.2.27}$$

The covariant equation of matter conservation in a Riemannian space tell us nothing about what is conserved, whereas the conservation law for the total tensor of energy-momentum $t^m r_n$ in the Minkowskian space tells us expressly that here we deal with the conservation of energy-momentum of matter and gravitational field. Thus, according to this theory, a Riemannian space emerges as a result of the action of a gravitational field on all forms of matter, therefore it is the effective Riemannian space of field origin. Minkowski's space finds its exact physical reflection in conservation laws for energy-momentum tensor and angular momentum of matter and gravitational field.

Since in a plane space there are ten Killing vectors, then there are ten conserved integral quantities for a closed system of fields.

The equation of conservation for the total tensor of energy-momentum in a Minkowskian space

$$D_m t_n^m = D_m (t_{gn}^m + t_{\mathrm{M}n}^m) = 0 \tag{3.2.28}$$

is equivalent to the covariant equation of matter conservation in a Riemannian space, and the latter is equivalent to the equations of motion for matter. Therefore, instead of the equations of motion for matter we can use (3.2.28).

It should be stressed that both matter and gravitational field are in this theory characterized by energy-momentum tensors, and so, unlike GTR, we here in principle do not deal with any pseudo-tensors and hence there are no nonphysical ideas that gravitational energy is nonlocalizable.

If we, following Hilbert and Einstein, would take the density of the Lagrangian of a gravitational field in a completely geometrized form, i.e., in a form that is only dependent on the metric tensor of a Riemannian space g^{ik} and their derivatives, e.g., $\mathfrak{L}_g = \sqrt{-g}R$, where R is the scalar curvature of the Riemannian space, then the density of the energy-momentum tensor for the free gravitational field in the Minkowskian

space will, by the field equations, always be equal to zero:

$$\frac{\delta \mathfrak{L}_g}{\delta \gamma^{mn}} = \frac{\delta \mathfrak{L}_g}{\delta g^{pk}} \frac{\partial g^{pk}}{\partial \gamma^{mn}} = 0. \tag{3.2.29}$$

Consequently, on the basis of a Minkowskian space, using a tensor physical field with an energy and momentum, it is in principle impossible to construct a completely geometrized Lagrangian of a gravitational field. Therefore, a theory formulated on the basis of a completely geometrized Lagrangian, in principle, cannot describe a Faraday–Maxwell physical gravitational field in Minkowski's space. In the literature it has been maintained (see, e.g., [50]) that a Minkowskian space, using a tensor field of spin 2, one can uniquely find the Lagrangian of the gravitational field of GTR, which is equal to the scalar curvature R. These works, however, have no physical content, since for the gravitational field introduced in them the energy-momentum tensor is zero, as follows from (3.2.29). Therefore, these works make no sense physically and their results are erroneous.

3.3. Main Identity

It was shown [51] that in a Minkowskian space the symmetrical tensor of the second rank f^{ik} can be decomposed into a direct sum of irreducible representations: one representation with spin 2, one with spin 1 and two with spin 0:

$$f^{lm} = [P_2 + P_1 + P_0 + P_{0'}]_{ik}^{lm} f^{ik}, \tag{3.3.1}$$

where P_s $(s = 2, 1, 0, 0')$ stand for the projection operators that meet the standard relations:

$$\begin{aligned} &P_s P^t = \delta_s^t P_t \quad \text{(here there is no summation over } t!) \\ &P^{in}_{s;in} = (2s + 1), \\ &\sum_s P^{lm}_{s;ik} = \frac{1}{2} (\delta_i^l \delta_k^m + \delta_i^m \delta_k^l) \equiv \delta_{ik}^{lm}. \end{aligned} \tag{3.3.2}$$

Operators P_s can conveniently be written at first in terms of momenta. We introduce the auxiliary (projection) quantities

$$X_{ik} = \frac{1}{\sqrt{3}} \left(\gamma_{ik} - \frac{q_i q_k}{q^2} \right), \qquad Y_{ik} = \frac{q_i q_k}{q^2}. \tag{3.3.3}$$

It can easily be shown that P_s, which obey (3.3.2), can in terms of

(3.3.3) be written as

$$P^{ml}_{0;ni} = X_{ni}X^{lm}, \quad P^{ml}_{0';ni} = Y_{ni}Y^{ml}, \tag{3.3.4}$$

$$P^{ml}_{1;ni} = \frac{\sqrt{3}}{2}\,[X^l_i Y^m_n + X^m_n Y^l_i + X^m_i Y^l_n + X^l_n Y^m_i], \tag{3.3.5}$$

$$P^{ml}_{2;ni} = \frac{3}{2}\,[X^l_i X^m_n + X^m_i X^l_n] - X_{ni}X^{ml}. \tag{3.3.6}$$

It is seen from (3.3.4)–(3.3.6) that the operators $P^{ml}_{s;nl}$ are symmetrical in indices (ml) and (ni).

In the x-representation the projection operators P_s are nonlocal integro-differential operators

$$(P^{ml}_{s;ni}f^{ni}) = \int d^4y P^{lm}_{s;ni}(x-y)f^{ni}(y).$$

Explicit expressions for $P^{lm}_{0;ni}(x)$ and $P^{ml}_{2;ni}$ have the form

$$P^{lm}_{0;ki}(x) = \frac{1}{3}\,[\gamma^{lm}\gamma_k\delta(x) + (\gamma^{lm}\partial_i\partial_k + \gamma_{ik}\partial^l\partial^m)D(x)$$

$$+ \partial_i\partial_k\partial^l\partial^m\Delta(x)], \tag{3.3.7}$$

$$P^{lm}_{2;ki}(x) = \left(\delta^{lm}_{ik} - \frac{1}{3}\gamma^{lm}\gamma_{ik}\right)\delta(x)$$

$$+\left[\frac{1}{2}(\delta^l_i\partial^m\partial_k + \delta^m_k\partial^l\partial_i + \delta^l_k\partial^m\partial_i + \delta^m_i\partial^l\partial_k)\right.$$

$$\left.-\frac{1}{3}(\gamma^{lm}\partial_i\partial_k + \gamma_{ik}\partial^l\partial^m)\right]D(x) + \frac{2}{3}\,\partial^l\partial^m\partial_i\partial_k\Delta(x). \tag{3.3.8}$$

By (3.3.7) and (3.3.8) $D(x)$ is Green's function of the wave equation

$$\Box D(x) = -\delta(x), \tag{3.3.9}$$

and

$$\Delta(x) = \int D(x-y)D(y)\,d^4y.$$

Therefore, it obeys the equation

$$\Box\Delta(x) = -D(x) \tag{3.3.10}$$

It can easily be checked from (3.3.7)–(3.3.10) that the operators P_0 and P_2 are conserved, i.e., for these operators we have the identities

$$\begin{aligned} \partial_l P^{lm}_{0;ni}(x) &= \partial^n P^{lm}_{0;ni}(x) \equiv 0, \\ \partial_l P^{lm}_{2;ni}(x) &= \partial^n P^{lm}_{2;ni}(x) \equiv 0. \end{aligned} \tag{3.3.11}$$

On the other hand, P_1 and $P_{0'}$ do not possess this property.

It is clear from (3.3.1) that if a tensor field obeys

$$\partial_l f^{lm} = 0, \tag{3.3.12}$$

it is not going to contain representations with spin 1 and 0′. This implies that such a tensor field describes only spins 2 and 0.

It can readily be shown that the operator

$$\Box(2P_0 - P_2) \tag{3.3.13}$$

is a unique, local and conserved operator of the second order. Operating with this on the function

$$\varphi^{in} - \frac{1}{2}\gamma^{in}\varphi,$$

where $\varphi = \gamma_{pq}\varphi^{pq}$, we will find from (3.3.7)–(3.3.10)

$$\Psi^{mn} = \partial_k\partial_p\,[\gamma^{nk}\varphi^{pm} + \gamma^{mk}\varphi^{pn} - \gamma^{kp}\varphi^{mn} - \gamma^{mn}\varphi^{kp}]\,. \tag{3.3.14}$$

The structure (3.3.14) for any symmetric tensor field is remarkable in that it is local, linear; it contains derivatives of the second order only, and satisfies a conservation law, i.e., the divergence of Ψ^{mn} is identically zero:

$$\partial_m\Psi^{mn} = 0. \tag{3.3.15}$$

In what follows we will need the structure (3.3.14), given in terms of derivatives covariant in Minkowski's metric for the density of the metric tensor $\tilde{g}^{lm}$:

$$J^{mn} = D_k D_p\,[\gamma^{np}\tilde{g}^{km} + \gamma^{pm}\tilde{g}^{kn} - \gamma^{kp}\tilde{g}^{mn} - \gamma^{mn}\tilde{g}^{kp}]. \tag{3.3.16}$$

It is obvious from (3.3.16) that

$$D_m J^{mn} \equiv 0. \tag{3.3.17}$$

We will call this identity the main identity, since it has a fundamental significance for RTG.

3.4. Equations of RTG

At the foundation of the relativistic theory of gravitation (RTG) [30], which has crowned the development of the ideas presented in [44], we have laid the following physical requirements:

(1) The theory must strictly satisfy the conservation laws for energy-momentum and angular momentum for matter and gravitational field. By matter we mean all forms of matter (including electromagnetic field) safe for gravitation. The conservation laws reflect the general dynamic

properties of matter and enable unified characteristics to be introduced for its various forms. The general dynamic properties of matter, given by the conservation laws, are represented through the structure of the geometry of space-time. It of necessity appears to be pseudo-Euclidean (Minkowski's space). Geometry is thus specified not by convention, as Poincaré believed, but is rather defined by the general dynamic behavior of matter, namely by the conservation laws. Minkowski's space possesses the four-parameter group of translations and the six-parameter group of rotations. This structure of space-time reflects the dynamic properties of matter – its conservation laws. This is the cardinal departure of RTG from GTR, and this extracts us from the tangle of Riemannian geometry.

(2) The gravitational field is described by a symmetrical tensor and is a real physical field that has an energy-momentum density. If we put into correspondence to this field some particles, they must have a zero rest mass. The real and virtual quanta of the gravitational field then have spin states 2 and 0.

This circumstance returns to the gravitational field its physical reality, since even locally it cannot be cancelled by any choice of a reference frame, and so there is no (even local) equivalence between the gravitational field and inertial forces. This physical requirement distinguishes RTG markedly from GTR.

In (1) and (2) we have introduced into gravitation the fundamental laws of conservation of energy-momentum and angular momentum, as well as the gravitational field of Faraday – Maxwell type that has an energy-momentum density.

Einstein in GTR identified gravitation with the metric tensor of a Riemannian space, but this resulted in the gravitational field no longer being a physical field. In addition, the fundamental conservation laws were lost. That is why we have to reject completely this point of Einstein.

(3) In terms of Minkowski's space and the concept of the physical gravitational field we formulate the principle of geometrization. It states that, owing to the property of universality, interaction of the gravitational field with matter is described by "combining" the tensor of the gravitational field with the metric tensor of a Minkowskian space. This can always be achieved since, whatever the form of matter, its original physical equations will include the metric tensor of Minkowski's space. There are no two ways about it, because physical processes proceed in time and space. Such a universal gravitational interaction is bound to give rise to an effective Riemannian space which has literally a field, dynamic origin. According to geometrization principle, the motion of matter under the action of a gravitational field in a Min-

kowskian space is identical to its motion in the effective Riemannian space. In 1921 Einstein discussed the structure of geometry: "The question of whether this continuum has Euclidean or Riemannian or any other structure is a physical question to be only answered by experiment, and not a question of convention on the choice from simple expediency." From the fundamental point of view, this statement of Einstein's is absolutely correct.

But things turned out to be much more complicated. The most important thing was to understand what physical properties of matter determine geometry. Indeed, if we define physical geometry from studies of the motion of light and test bodies, we can assume that we thereby have established the Riemannian structure of geometry. Does this imply that we have thus to use this geometry as a basis of our theory? No, it doesn't, because if we did so, this would automatically divest us of the fundamental laws of conservation of energy-momentum and angular momentum, because geometry does not possess a group of motion of space-time.

This all has happened in GTR. Accordingly, even having discovered experimentally a Riemannian geometry, we should not jump to conclusions concerning the structure of geometry that has to be laid at the foundation of the theory. We should first of all clarify whether this concept is primary, of perhaps it is secondary in origin. In doing so, we should start off with the general dynamic properties of matter – its conservation laws, because it is these laws that are those guiding principles that are to be used to blaze the trail to a physical theory.

In other words, not some particular physical manifestations of the motion of matter, but rather the most general dynamic properties define the structure of a physical geometry underlying a theory. In our theory (RTG) physical geometry is defined not from studies of the motion of light and test bodies, but on the basis of the general dynamic properties of matter – its conservation laws, which are fundamentally valuable and experimentally tested. In this picture, the motion of light and test bodies stems from the simple action of a gravitational body on matter in a Minkowskian space.

The Minkowskian space and the gravitational field are original primary concepts, and the effective Riemannian space is a secondary concept, one that owes its origin to the gravitational field and its universal action on matter. The very essence of geometrization principle includes the separation of inertial forces and gravitational fields. But this separation can only be realized physically when the equations for the gravitational field include the metric tensor of Minkowski's space. In GTR, as is easily seen from the Hilbert – Einstein equations, this separation is impossible. In RTG Minkowski's space is reflected not only in the

conservation laws, but also in the way physical phenomena are described.* Therefore, Minkowski's space is a physical one, and hence observable. Its characteristics can always be checked by appropriately processing experimental evidence for the motion of light signals and test bodies in the "effective" Riemannian space. V. Fock wrote: "As to the consideration that the straight line, like a light beam, is more directly observable, it makes no sense: in definitions a decisive factor is not direct observability, but correspondence to nature, even if this correspondence has been established by indirect mental constructs." Observability should thus be understood not in the primitive, but in a more general and profound sense as adequacy to nature. To be sure, RTG in no way excludes the possibility of describing motion in the effective Riemannian space.

The equations of RTG (unlike GTR) include the metric tensor of the Minkowskian space, therefore all functions describing physical fields are written in terms of unified coordinates for the entire Minkowskian space-time, e.g., in Galilean (Cartesian) coordinates. When combined with the field equations, which define the structure of the gravitational field, they give rise to an absolutely new physical meaning of the Hilbert – Einstein equations, which are also changed and simplified in the process.

The conservation laws for matter plus gravitational field are consequences of the RTG equations and they reflect the pseudo-Euclidean structure of space-time. Solving the system of the field equations, we will establish the dependence of the metric tensor of the effective Riemannian space both on the coordinates of the Minkowskian space and on the gravitational constant G. Proper times measured by a clock (that moves with the matter) turns out to be dependent on the coordinates of Minkowski's space and the gravitational constant G. The course of proper time is thus dependent on the nature of the gravitational field.

The presence of Minkowski's space and of its metric tensor in the field equations enables inertial forces to be separated from gravitational field and its influence on some other physical processes to be revealed. On the other hand, all the variables of the field can be written in common coordinates for the entire space-time, e.g., such coordinates can

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*The unique and profound connection of the conservation laws (which have been tested experimentally) with the structure of Minkowski's space attests to its physical observability.

Experiments on the propagation of light signals and test bodies only tell us something about the effective Riemannian space, which has emerged due to the action of a gravitational field on matter according to geometrization principle. GTR cannot contain the concepts of Minkowski's space, and so it simply makes no sense speaking about it.

Galilean (Cartesian). These features are in principle absent in GTR, since in the Riemannian geometry there are no global Cartesian coordinates. Within the framework of GTR we cannot write the Hilbert – Einstein equations in the coordinates of Minkowski's space, since in the Riemannian geometry, on which GTR is based, there is no such concept.

In this section we will construct, within the framework of the special theory of relativity and the principle of geometrization, the relativistic equations for matter and gravitational field.

We can always choose the simplest connection between the effective metric of the field Riemannian space and the gravitational field:

$$\tilde{g}^{ik} = \sqrt{-g}\, g^{ik} = \sqrt{-\gamma}\, \gamma^{ik} + \sqrt{-\gamma}\, \varphi^{ik}. \tag{3.4.1}$$

The field variable of the gravitational field in our theory is the tensor φ^{ik}. We will consider that in the general case the gravitational field only has spins 2 and 0. As we have seen in Section 3.2, such physical requirements in Galilean coordinates yield the following four equations of the gravitational field:

$$\partial_i \varphi^{ik} = \partial_i \tilde{g}^{ik} = 0. \tag{3.4.2}$$

Similar conditions have sometimes been used earlier [9, 52] in GTR as a special class of coordinate harmonic conditions in dealing with island-type problems. The importance of the harmonic coordinate conditions for solving island-type problems was especially noted by Fock [9]. So, he wrote: "The above comments concerning the privileged nature of the harmonic coordinate system should by no mean be taken to mean that there is some ban on the uses of other coordinate systems. Nothing can be further from our viewpoint than such interpretation of it. . ."; he goes on to say that ". . . the existence of harmonic coordinates, although a fact of primary theoretical and practical importance, by no means excludes possibilities of using other, nonharmonic, systems of coordinates." In terms of our theory, in dealing with island-type problems Fock was unaware dealing with conventional Galilean coordinates in an inertial reference frame, and these coordinates, as we know from the special theory of relativity, are, of course, distinguished. In Fock's arguments the harmonic conditions were therefore not coordinate conditions, as he believed, but, as we will see later in the book, from our theory, they were the field equations in Galilean coordinates of an inertial reference frame. That is why they have played so important role in specific calculations, a fact of which Fock, and others for that matter, has not suspected.

In consequence, Fock viewed harmonic conditions only as privileged coordinate conditions, only for island-type problems at that. This

is quite understandable: he, like all of his great predecessors, remained in the captivity of Riemannian geometry, and the latter did not allow a more profound penetration into the essence of the problem. To make a radical step and put forward these conditions it was necessary to forsake the ideology of GTR, to get out of the hold of Riemannian geometry, to extend, despite GTR, the special principle of relativity to cover gravitational phenomena, and to introduce the ideas that the gravitational field is a Faraday – Maxwell type physical field, with its energy and momentum. These features are all incorporated into our theory; the coordinate system is chosen arbitrarily, it is only specified by the metric tensor γ^{ik} of Minkowski's space, as it is customary in the theory of elementary particles. On the other hand, equations (3.4.2) in our theory are universal, because they are equations of the gravitational field. They have nothing to do with our choice of a coordinate system. In Minkowski's space these equations are written in covariant form

$$\sqrt{-\gamma}\, D_i \varphi^{ik} = D_i \tilde{g}^{ik} = 0. \qquad (3.4.3)$$

We see from Section 3.3 that these field equations automatically exclude from the gravitational tensor field spins 1 and 0′. For the desired fourteen variables of the gravitational field and matter we have thus available the four covariant equations (3.4.3). To construct the following ten equations we will use the simple but far-reaching analogy of the electromagnetic field. Since any vector field A^n contains spin 1 and spin 0, it can be expanded as a direct sum of appropriate irreducible representations. This expansion can be realized using the projection operators (3.3.3) introduced in Section 3.3:

$$A^n = X^n_m A^m + Y^n_m A^m. \qquad (3.4.4)$$

The operator X^n_m is conserved here, i.e., it obeys

$$\partial_n X^n_m = \partial^m X^n_m = 0. \qquad (3.4.5)$$

and the operator Y^n_m has no such property.

It is well known from electrodynamics that the source of an electromagnetic field A^n is the conserved electromagnetic current j^n. It is quite natural therefore to employ for the derivation of the equation of motion of the field another conserved operator – X^n_m. But this operator is nonlocal in nature. Using it, however, we can construct the only local, linear and conserved operator $\Box X^n_m$, which contains only second derivatives. Operating with it on A^m, we will obtain an expression, which in terms of covariant derivatives, will have the form

$$\gamma^{mk} D_m D_k A^n - D^n D_m A^m. \qquad (3.4.6)$$

Postulating the relation

$$\gamma^{mk} D_m D_k A^n - D^n D_m A^m = \frac{4\pi}{c} j^n, \qquad (3.4.7)$$

we will obtain Maxwell's equations.

One important feature of the equation of electrodynamics (3.4.7) is its being invariant under the following gauge transformation:

$$A^n \to A^n + D^n \varphi, \qquad (3.4.8)$$

where φ is an arbitrary scalar function.

All physical quantities remain unchanged under the gauge transformation (3.4.8). This implies that they are independent of the presence of spin 0 in the vector field A^n. Therefore, we can take a gauge transformation such that it would eliminate spin 0 from the vector field for ever. The latter implies that we can introduce the condition

$$D_m A^m = 0. \qquad (3.4.9)$$

We may thus introduce the condition (3.4.9) in electrodynamics; although we may introduce no such condition, for spin 0 of a vector field does not affect the physical quantities because of the gauge invariance.

From (3.4.0) and (3.4.7), we will find the system of equations

$$\gamma^{mk} D_m D_k A^n = \frac{4\pi}{c} j^n, \quad D_m A^m = 0,$$

which define the vector-potential A^n, which has only spin 1.

The Lagrangian formalism, which yields these results, is well known. Notice that the idea of constructing a theory of interaction for vector fields (both Abelian and non-Abelian) on the basis of gauge invariance turned out to be quite productive and it is being elaborated on at present.

Problems to be encountered while constructing the remaining equations for the tensor gravitational field have an absolutely different nature, since the source of the field – the tensor of energy-momentum – is invariant under gauge transformations of the field $\tilde{\varphi}^{ik}$. By analogy with Maxwell's electrodynamics, we will formulate other equations for the tensor gravitational field. The only conserved tensor of the second rank is the tensor of energy-momentum of matter and of gravitational field in a Minkowskian space t^{mn}, and so it can naturally be taken as the complete source of the gravitational field. We have found in Section 3.3 that the simplest, identically conserved tensor linear in $\tilde{g}^{mn}$ is the quantity J^{mn}, therefore, by analogy with electrodynamics,

we will postulate the following equation:

$$J^{mn} \equiv D_k D_p [\gamma^{kn} \tilde{g}^{pm} + \gamma^{km} \tilde{g}^{pn} - \gamma^{kp} \tilde{g}^{mn} - \gamma^{mn} \tilde{g}^{kp}]$$
$$= \lambda(t_{\mathrm{g}}^{mn} + t_{\mathrm{M}}^{mn}). \tag{3.4.10}$$

This type of equation, generally speaking, presupposes that the conservation law for the tensor of energy-momentum of matter and gravitational field is automatically met in the Minkowskian space, i.e.,

$$D_m (t_{\mathrm{g}}^{mn} + t_{\mathrm{M}}^{mn}) \equiv D_m t^{mn} = 0. \tag{3.4.11}$$

As a consequence (see (3.2.28)), also holds the covariant law of matter conservation in a Riemannian space

$$\nabla_m T^{mn} = 0. \tag{3.4.12}$$

The Hilbert tensor of energy-momentum T^{mn} can be defined phenomenologically. Then (3.4.12) will be the equations of motion for matter. Using (3.4.3) and (3.4.10) gives

$$\gamma^{kp} D_k D_p \tilde{g}^{mn} = -\lambda(t_{\mathrm{g}}^{mn} + t_{\mathrm{M}}^{mn}), \tag{3.4.13a}$$

$$D_m \tilde{g}^{mn} = 0. \tag{3.4.13b}$$

The simultaneous equations (3.4.13) are exactly the system we seek for the relativistic theory of gravitation.

The role of equations (3.4.13b) in RTG differs markedly from that of (3.4.9) in electrodynamics. Indeed, although the left-hand side of equations (3.4.10) is invariant under the gauge transformation

$$\tilde{g}^{mn} \to \tilde{g}^{mn} + D^m \tilde{\xi}^n + D^n \tilde{\xi}^m - \gamma^{mn} D_k \tilde{\xi}^k, \tag{3.4.14}$$

where $\tilde{\xi}^n = \sqrt{-\gamma}\, \xi^n$ is the density of an arbitrary four-vector $\xi^n(x)$ due to the fact that the right-hand side of (3.4.10) is noninvariant under the change (3.4.14), we do not have in the theory an arbitrariness of the type (3.4.14), and so equations (3.4.3) cannot be a consequence of equations (3.4.10).

In RTG equations (3.4.3) are thus additional independent dynamic equations of the gravitational field, rather than some coordinate or gauge conditions.

When formulating the theory the main question to be answered is whether there exists the Lagrangian density for the gravitational field with spins 2 and 0 that would automatically lead, by the least-action principle, to equations (3.4.13).

The general Lagrangian density of the gravitational field $\tilde{\varphi}^{ik}$ that describes spins 2 and 0 and is quadratic in the first derivatives of the

field has the form

$$\mathfrak{L}_g = a\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{kq}D_p\tilde{g}^{mn} + b\tilde{g}_{kq}D_m\tilde{g}^{pq}D_p\tilde{g}^{km}$$
$$+ c\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{km}D_p\tilde{g}^{nq}. \tag{3.4.15}$$

A characteristic feature of the Lagrangian is the fact that the convolution of covariant derivatives with respect to the Minkowski metric is done using the effective metric tensor $\tilde{g}^{ik}$ of a Riemannian space. It can be shown that this requirement for the gravitational field is a consequence of geometrization principle and the structure of the gravitational field that only has spins 2 and 0.

According to the least-action principle, the system of equations for the gravitational field assumes the form

$$\frac{\delta\mathfrak{L}_g}{\delta\tilde{\varphi}^{ik}} + \frac{\delta\mathfrak{L}_M}{\delta\tilde{\varphi}^{ik}} \equiv \frac{\delta\mathfrak{L}_g}{\delta\tilde{g}^{ik}} + \frac{\delta\mathfrak{L}_M}{\delta\tilde{g}^{ik}} = 0. \tag{3.4.16}$$

This takes into account the constraint (3.4.1). In (3.4.16) $\mathfrak{L}_M$ is the Lagrangian density for matter, and the Lagrangian density $\mathfrak{L}_g$ is given by (3.4.15).

For the system (3.4.16) to be representable in the form (3.4.13), in (3.4.15) we will have to select the constants *a, b,* and *c* in a definite and unique manner. To this end, from (3.2.17), (3.2.22), and (3.2.25), we will find for the Lagrangian $\mathfrak{L} = \mathfrak{L}_g + \mathfrak{L}_M$ the density of the energy-momentum tensor for matter and gravitational field t^{mn} in Minkowski's space. Taking the variation of the total Lagrangian with respect to γ_{mn}, we will obtain

$$t^{mn} = 2\sqrt{-\gamma}\left(\gamma^{nk}\gamma^{mp} - \frac{1}{2}\gamma^{mn}\gamma^{pk}\right)\frac{\delta\mathfrak{L}}{\delta\tilde{g}^{kp}} + 2bJ^{mn}$$
$$+ D_p\{(2a+b)[H_k^{pn}\gamma^{km} + H_k^{pm}\gamma^{kn} - H_k^{mn}\gamma^{kp}]$$
$$- 2(a+2c)\gamma^{mn}\tilde{g}^{kp}\tilde{g}_{lq}D_k\tilde{g}^{lq}\}, \tag{3.4.17}$$

where

$$H_k^{pn} = (\tilde{g}^{pl}D_l\tilde{g}^{qn} + \tilde{g}^{nl}D_l\tilde{g}^{pq})\tilde{g}_{qk}.$$

We see from (3.4.17) that the equations

$$t^{mn} = 2bJ^{mn} + D_p\{(2a+b)[H_k^{pn}\gamma^{km} + H_k^{pm}\gamma^{kn} - H_k^{mn}\gamma^{kp}]$$
$$- 2(a+2c)\gamma^{mn}\tilde{g}^{kp}\tilde{g}_{lq}D_k\tilde{g}^{lq}\} \tag{3.4.18}$$

are equivalent to the field equations (3.4.16).

In order that the relation

$$D_m t^{mn} = 0 \tag{3.4.19}$$

may not give rise to any new equation for the field φ^{ik}, which would lead to an overdetermined system of equations, it is necessary and sufficient for the coefficients *a, b,* and *c* to meet the conditions

$$a = -\frac{1}{2}b, \qquad c = \frac{1}{4}b. \tag{3.4.20}$$

With such a choice of the constants we have

$$D_m t^{mn} = 0.$$

The equations of motion of matter thus follow directly from the equations for the gravitational field. From (3.4.20), the expression (3.4.18) will take the form

$$D_p D_k(\gamma^{km}\tilde{g}^{pn} + \gamma^{kn}\tilde{g}^{pm} - \tilde{g}^{mn}\gamma^{kp} - \gamma^{mn}\tilde{g}^{kp}) = \frac{1}{2b}(t_{\text{g}}^{mn} + t_{\text{M}}^{mn}) \equiv \frac{1}{2b}t^{mn}, \tag{3.4.21}$$

which coincides with the equations (3.4.10) we have written earlier, by analogy with electrodynamics, if we put

$$2b = \frac{1}{\lambda}.$$

It follows that the Lagrangian density $\mathfrak{L}_{\text{g}}$ that leads to the field equations in the form (3.4.21) is

$$\mathfrak{L}_{\text{g}} = \frac{1}{2\lambda}\left[\tilde{g}_{kq}D_m\tilde{g}^{pq}D_p\tilde{g}^{km} - \frac{1}{2}\tilde{g}_{km}\tilde{g}_{nq}\tilde{g}^{lp}D_l\tilde{g}^{kq}D_p\tilde{g}^{mn} + \frac{1}{4}\tilde{g}_{km}\tilde{g}_{nq}\tilde{q}^{lp}D_l\tilde{g}^{km}D_p\tilde{g}^{nq}\right]. \tag{3.4.22}$$

According to the principle of conformity, the constant λ is

$$\lambda = -16\pi. \tag{3.4.23}$$

The Lagrangian density (3.4.22) can, from (3.4.23), be represented as

$$\mathfrak{L}_{\text{g}} = \frac{1}{32\pi}[\tilde{G}^l_{mn}D_l\tilde{g}^{mn} - \tilde{g}^{mn}\tilde{G}^k_{mk}\tilde{G}^l_{nl}] \tag{3.4.24}$$

where the tensor of rank three $\tilde{G}^k_{lm}$ is given by

$$\tilde{G}^k_{lm} = \frac{1}{2}\tilde{g}^{pk}(D_m\tilde{g}_{lp} + D_l\tilde{g}_{mp} - D_p\tilde{g}_{lm}). \tag{3.4.25}$$

Also, we can write $\mathfrak{L}_g$ as

$$\mathfrak{L}_g = -\frac{1}{16\pi}\sqrt{-g}\,g^{mn}\,[G^k_{lm}G^l_{nk} - G^l_{mn}G^k_{lk}]. \tag{3.4.26}$$

This form of the Lagrangian was first considered by Rosen [53]. In (3.4.26) the tensor of rank three G^k_{ml} is

$$G^k_{ml} = \frac{1}{2}\,g^{kp}(D_m g_{pl} + D_l g_{pm} - D_p g_{lm}). \tag{3.4.27}$$

From (3.4.3) the complete system of equations for matter and gravitational field will be [28-30]

$$\gamma^{pk}D_p D_k \tilde{g}^{mn} = 16\pi t^{mn}, \tag{3.4.28}$$

$$D_m \tilde{g}^{mn} = 0. \tag{3.4.29}$$

Clearly, in a Galilean coordinate system the equations (3.4.28)–(3.4.29) become

$$\Box\,\tilde{g}^{mn} = 16\pi t^{mn}, \tag{3.4.28$'$}$$

$$\partial_m \tilde{g}^{mn} = 0. \tag{3.4.29$'$}$$

If we limited ourselves to the equations (3.4.21), the division of the metric of a Riemannian space into the metric in a Minkowski space and a tensor gravitational field would be conditional in nature and would be devoid of physical meaning. The second system (3.4.29) of four field equations in principle separates everything that is concerned with inertial forces from everything that is concerned with the gravitational field. Both systems of equations (3.4.28) and (3.4.29) are generally covariant. The behavior of a gravitational field is usually subject to appropriate physical conditions from a specified, e.g., Galilean system of coordinates. In GTR it is impossible to formulate the physical conditions imposed on the metric g^{mn}, while remaining in a Riemannian space, since the asymptotic behavior of the metric is always dependent on our choice of a three-dimensional coordinate system.

We now want to find the explicit form of the system (3.4.16). We can show that the Lagrangian (3.4.22) obeys

$$\frac{\partial\mathfrak{L}_g}{\partial\tilde{g}^{mn}} = \frac{1}{16\pi}\,[G^k_{ml}G^l_{kn} - G^k_{mn}G^l_{kl}]$$

and

$$\frac{\partial\mathfrak{L}_g}{\partial\tilde{g}^{mn}_{\ \ ,k}} = \frac{1}{16\pi}\left[G^k_{mn} - \frac{1}{2}\delta^k_m G^l_{nl} - \frac{1}{2}\delta^k_n G^l_{ml}\right].$$

Therefore,

$$\frac{\delta \mathfrak{L}_{\rm g}}{\delta \tilde{g}^{mn}} = -\frac{1}{16\pi} R_{mn}, \tag{3.4.30}$$

where R_{ml} is the curvature tensor of rank two of the Riemannian space, which is

$$R_{mn} = D_k G^k_{mn} - D_m G^l_{nl} + G^k_{mn} G^l_{kl} - G^k_{ml} G^l_{nk}. \tag{3.4.31}$$

Since, by (3.2.6b) and (3.2.11),

$$2 \cdot \frac{\delta \mathfrak{L}_{\rm M}}{\delta g^{mn}} = \frac{1}{\sqrt{-g}}\left(T_{mn} - \frac{1}{2} g_{mn} T\right), \tag{3.4.32}$$

then, by (3.4.16),

$$\sqrt{-g}\, R_{mn} = 8\pi\left(T_{mn} - \frac{1}{2} g_{mn} T\right). \tag{3.4.33}$$

We have thus arrived at the well-known system of equations of Hilbert–Einstein with the only substantial difference that all the variables of the field in them are functions of the coordinates of the Minkowski space, whose metric tensor enters into the equations (3.4.29). Physical significance is only inherent in solutions such that under appropriate initial and boundary conditions they satisfy both (3.4.28) and (3.4.29).

The joint system (3.4.28) and (3.4.29) is complete if it contains as many equations as there are unknown variables of the field. The Hilbert–Einstein equations (3.4.33) do not contain the metric tensor of a Minkowski space, and so introducing this tensor is an illusion. Only equations (3.4.29) enable inertial forces to be separated uniquely from the gravitational field and thereby they enable us to introduce into the theory a plane Minkowskian space. We stress that equations radically change the nature of the solution of the Hilbert–Einstein equations, leading to new physical predictions. This all is accomplished in RTG. The Lagrangian density of the gravitational field (3.4.22) is the only one that yields a self-consistent system of equations for field and matter (3.4.28) and (3.4.29). This means that the equations of RTG are the only simple equations of the second order.

Because of the importance of this fact we will also provide another form of the proof of the equivalence of equations (3.4.10) and (3.4.33), based on direct calculation of the tensor densities $t_{\rm g}^{mn}$ and $t_{\rm M}^{mn}$ in the Minkowski space.

Using formulas (3.2.17) and the constraint (3.4.1), we will find that the density of the tensor of the energy-momentum of the gravitational

field in the Minkowski space for the Lagrangian density (3.4.22) is

$$t_{\mathrm{g}}^{mn} = -\frac{1}{16\pi} J^{mn} - \frac{\sqrt{-\gamma}}{8\pi}\left(\gamma^{mp}\gamma^{nk} - \frac{1}{2}\gamma^{mn}\gamma^{pk}\right)R_{pk}. \quad (3.4.34)$$

We see thus that we have automatically gotten here the curvature tensor of the second rank for a Riemannian space R_{pk}. Likewise, using (3.2.17) and (3.4.1), and also the definition of the Hilbert tensor density (3.2.6a) for the density of the energy-momentum tensor of matter in the Minkowski space, we will get

$$t_{\mathrm{M}}^{mn} = \sqrt{\frac{\gamma}{g}}\left(\gamma^{mp}\gamma^{nk} - \frac{1}{2}\gamma^{mn}\gamma^{pk}\right)\left(T_{pk} - \frac{1}{2}g_{pk}T\right). \quad (3.4.35)$$

Substituting (3.4.34) and (3.4.35) into the field equations (3.4.10), we will obtain

$$\left(\gamma^{mp}\gamma^{nk} - \frac{1}{2}\gamma^{nm}\gamma^{pk}\right)\left[R_{pk} - \frac{8\pi}{\sqrt{-g}}\left(T_{pk} - \frac{1}{2}g_{pk}T\right)\right] = 0.$$

From this we arrive at the system of equations for the gravitational field in the form (3.4.33).

The system of equations (3.4.10) is thus equivalent to the Hilbert–Einstein system of equations (3.4.33). But the complete system of equations for matter and gravitational field (3.4.28) and (3.4.29) is equivalent to the system

$$\sqrt{-g}\,R_{mn} = 8\pi\left(T_{mn} - \frac{1}{2}g_{mn}T\right), \quad (3.4.36)$$

$$D_m\tilde{g}^{mn} = 0. \quad (3.4.37)$$

It should be stressed once more that the equations (3.4.37) are general and universal, since these are field equations that describe gravitational fields with spins 2 and 0. The choice of a reference frame (or a coordinate system) is determined by the metric tensor of the Minkowski space. On the other hand, equations (3.4.27) impose no constraints on the choice of a coordinate system. Thus, equations (3.4.37) exclude from the density of the tensor field $\tilde{\varphi}^{ik}$ spins 1 and 0′, which leaves us only with spins 2 and 0. The required six components of the gravitational field that correspond to spins 2 and 0 and the four components of matter are found from the field equations (3.4.28) or equivalent Hilbert–Einstein equations (3.4.36).

Notice that some aspects of the theory of gravitation in the Minkowski space have been considered in works [53-55]. But even those

among these authors who at one time had been on the right track could not understand this and went in another direction to end up with a theory of gravitation that was far from a complete one. Our works [28-30] complete the construction of the relativistic theory of gravitation that leads to a number of new important predictions.

We now turn to some physical consequences of RTG. It is well known that in terms of Friedmann's homogeneous and isotropic Universe there may, by GTR, be three models of the Universe. One of them is a closed Universe, having a finite volume. What is the density of matter in the present Universe? GTR cannot give an answer to this question. According to RTG, the homogeneous and isotropic Universe of Friedmann is infinite and only "plane", since its three-dimensional geometry is Euclidean. Correspondingly, the density of matter energy in the Universe must be equal to the critical density determined from measurements of Hubble's constant. RTG predicts that the Universe must contain "latent mass", such that its energy density is nearly 40 times that of the density of matter energy observed today. Another important consequence of RTG is the statement that the total density of matter energy and gravitational field in the Universe must add up to zero.

We see that RTG's predictions for the evolution of Friedmann's homogeneous and isotropic Universe differ substantially from the conclusions of GTR.

Further, it follows from GTR that massive objects with a mass larger than three solar masses during a finite span of proper time are compressed indefinitely by gravitational forces till they reach infinite density. This evolutional process is called the gravitational collapse. Objects of that sort are known as "black holes". They have no material surface, and so a body that gets into a "black hole" and crosses its boundary will encounter nothing but empty space. From the inside of the "black hole" even light cannot escape through its boundary. Wheeler regarded gravitational collapse and the resultant singularity as "one of the greatest crises of all time" for fundamental physics. Anyone who has perceived the essence of GTR will agree to that. The relativistic theory of gravitation changes radically the character of gravitational collapse. It predicts the gravitational slowing of time, owing to which a massive body in the co-moving reference frame is compressed during a finite proper time and, most important, the matter density remains finite and cannot be higher than 10^{16} g/cm^3, the brightness of the body decays exponentially, the object "blackens" but, unlike a "black hole", always has a material surface. Such objects, if they emerge, are complex in structure, although there is no gravitational self-closure in the process. Matter thus does not disappear from our space. In terms of RTG proper time for a falling test body varies

both with the coordinates of the Minkowskian space and with the gravitational constant G. Accordingly, the flow of proper time is determined by the nature of the gravitational field. It is this circumstance that is responsible the infinite slowing of proper time for the falling body as it approaches the Schwarzschild radius. It follows that, according to RTG, in nature there can be no "black holes" in which matter would be compressed catastrophically to infinite density and which would have no material surface. This all can be established with fair accuracy by taking as an example a spherically symmetric nonstationary problem for dust, when the pressure is assumed to be zero. A period of proper time $d\tau$ for a falling body is related to the period of time dt in the Minkowski space by

$$d\tau = dt\left(\frac{\rho - GM}{\rho + GM}\right),$$

where ρ is a radial variable in the Minkowski space.

It follows immediately from this formula that when ρ approaches GM, the proper time increment $d\tau$ tends to zero, and so all physical processes for the falling body slow down indefinitely. According to RTG, there do not exist not only static but also nonstatic spherically symmetrical bodies with radius less or equal to GM. This means that there can be no holes in space-time. These pronouncements set RTG predictions apart from those of GTR. Massive objects will, of course be compressed less, when the pressure is nonzero, since the pressure is nonzero, since the pressure hinders gravitational attraction. The evolution of real objects calls for more detailed studies using the equations of state of matter, and it is a fascinating problem.

RTG explains the entire body of available observational and experimental evidence concerned with gravitational effects in the Solar System. Detailed analysis shows that GRT predictions are ambiguous for gravitational effects in the Solar System. So for some effects arbitrariness occurs in first-order terms in G, for others in second-order terms. Where does the reason for this ambiguity lie? In GTR to define the metric of a Riemannian space in some coordinates it is necessary to specify the so-called coordinate conditions, which are quite arbitrary and always noncovariant (i.e., they only refer to the coordinate system we have chosen). Depending on the kind of these conditions we in the same coordinates in the general case will by all means obtain different metric tensors. But different metric tensors in the same coordinates will also yield different geodesics, and so the predictions of GTR for the motion of light and test bodies will also be different.

Note that Weyl and Lorentz indicated that, given the equations for all time-like and all isotropic geodesics in some coordinate system,

the metric tensor of space-time in this system is determined up to a constant, i.e., on the physical side, when we study the motion of light and test bodies we can experimentally establish the structure of the geometry of space-time. According to this theorem, different metric tensors in a given coordinate system yield different predictions concerning the motion of light and test bodies.

Consider one hypothetic experiment, which clearly demonstrates the ambiguity of GTR predictions. In an inertial reference frame we have two test bodies fixed at different points A and B, and at point O lying close to line AB and equidistant from A and B, a "needle" is fixed onto which we can fix a massive (small-size) body M. Using this arrangement we will conduct two experiments. At first, we will take body M away from the arrangement at a distance that is much larger than AB (at infinity) and then time the propagation of a light signal from A to B, and back. We next bring body M back, put it on the needle and repeat the timing. In the presence of body M instead of t_0 we will have t, and their difference will give the time of gravitational delay $t - t_0 = \Delta t$ due to the action of the body on the motion of the light signal. If now we, in the second experiment, calculate the propagation time t (using, say, in the same coordinates the harmonic and Schwarzschild solutions) and then subtract the result t_0, the time delay for such different solutions in the same coordinates will appear to be different*. We are thus led to conclude that GTR gives no definite prediction for the given experiment.

We next pass on the gravitational radiation. In a paper on gravitational waves [8] Einstein wrote: "One might suppose that by adequately selecting a reference system one can always provide that all the components of the energy of the gravitational field would vanish, which would be of great interest. It can readily be shown, however, that, generally speaking, this is not the case." Einstein, in accordance with his principle of conformity, expected that all the components of the "energy of the gravitational field" vanish, therefore he considered this to be of great interest. He could not establish this, however. It was not until fairly recently that the gravitational radiation, as defined in GTR by Einstein, was shown indeed to be reducible to zero by an adequate choice of a permissible reference system. This is precisely the result that interested Einstein so much. And so the last phrase

*The multivaluedness of solutions of the Hilbert–Einstein equations in the same coordinates has, to put it mildly, escaped the attention of Academician Zeldovich, and so his opinion (*Usp. Fiz. Nauk,* 1986, **149**, 4) that the description is unique is simply erroneous. Zeldovich's paper also contains other wrong statements, but they will be discussed elsewhere.

in the above quotation is wrong. But if we can do away with the radiation, while still in the framework of GTR, this suggests that Einstein's formula for the quadruple gravitational radiation is no consequence of his theory. Here Einstein was rather guided by his formidable intuition than by the logic of his theory. It helped him to obtain a correct formula for the radiation but did not allow to reveal the essence of GTR. In RTG the gravitational field is a physical field and it cannot in principle, even locally, be "killed" by an adequate choice of a reference frame. RTG predicts the existence of gravitational waves that transport energy and momentum, which is in principle absent in GTR. The statement that GTR predicts the existence of gravitational waves is simply erroneous and comes from a lack of understanding of the logic of the theory. Einstein's formula is a consequence of the relativistic theory of gravitation, not GTR.

To sum up: from the conservation laws and the views on the gravitational field as a physical field that possesses an energy-momentum density, in combination with the principle of geometrization and local gauge invariance, we have uniquely formulated a relativistic theory of gravitation that explains all the available observational and experimental evidence for gravitation and makes predictions concerning the evolution of Friedmann's Universe and the gravitational collapse.

3.5. On Ambiguity of GTR Predictions for Gravitational Effects and Fundamentals of RTG

GTR has no conservation laws, and this is not its only fundamental drawback. As we will see later, ambiguity is inherent in GTR and concerns all gravitational effects. We will illustrate this by the example of the effect of the gravitational delay of a radio signal in the field of a static centrally symmetric body of mass M.

Starting off with the Hilbert–Einstein equations (3.4.36) we should take into account the fact that the space-time coordinates x^μ incorporated into this equation represent a manifold fixed by the arithmetization of space-time chosen. It will, therefore, be no loss of generality, if making convention about the arithmetization of space, we will put into correspondence, in variables $x^\mu = (t, r, \theta, \varphi)$, to centre S of a body M the point $r = r_S = 0$; to any point on the surface of the body (taking it to be a ball) a value $r = r_f$; to the location of a radio pulse source a point $e(r_1, \varphi_1, \theta_1 = \pi/2)$; and to the position of the receiver or the reflector that reflects the signals back to point e, a point $p(r_2, \varphi_2, \theta_2 = \pi/2)$. In the arithmetization chosen, one general external (in relation to body M) solution of the Hilbert–Einstein equations

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G T_M^{\mu\nu},$$

(where $R^{\mu\nu}$ is the Ricci tensor, $R = R^{\mu\nu}g_{\mu\nu}$) will (outside of the body) be a solu-

tion [56] with an arbitrary function $C(r)$

$$g_{kn} = -C\delta_{kn} + (C-A)\frac{x^k x^n}{r^2}, \qquad g_{00} = B,\ g = -BAC^2,$$

$$g^{kn} = -\frac{\delta_{kn}}{C} + \left(\frac{1}{C} - \frac{1}{A}\right)\frac{x^k x^n}{r^2}, \qquad g^{00} = \frac{1}{B},$$

where

$$B = 1 - \frac{2GM}{r\sqrt{C}}, \qquad A = C\left(1 + \frac{rC'}{2C}\right)^2 \left(1 - \frac{2GM}{r\sqrt{C}}\right)^{-1},$$

and $C' = \partial C/\partial r$. As to the function $C(r)$, it is only required that it be smooth and $\lim\limits_{r\to\infty} C(r) \to 1$.

It is this arbitrariness that eventually leads to ambiguous predictions of GTR for gravitational effects (and an uncertain energy-momentum of the gravitational field (GF)) in the field of a centrally symmetric source. Indeed, if we take the example

$$C(r) = 1 \quad \text{or} \quad C(r) = (1 + GM/r)^2, \tag{3.5.1}$$

we will obtain two different particular solutions for $g_{\mu\nu}(r, GM)$ that define the element ds^2

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{3.5.2}$$

in the first case and

$$ds^2 = \left(\frac{r-GM}{r+GM}\right) dt^2 - \left(\frac{r+GM}{r-GM}\right) dr^2 - r^2\left(1 + \frac{GM}{r}\right)^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{3.5.3}$$

in the second. In both solutions the coordinate r is the same (just as t, θ, φ), i.e., both in (3.5.2) and (3.5.3) corresponding to points $r = r_S = 0$, $r = r_f$, $e(r_1, \varphi_1, \theta_1 = \pi/2)$ and $p(r_2, \varphi_2, \theta_2 = \pi/2)$ are the positions of the centre S of the body M, its surface, source and receiver (or reflector) of radio pulses.

Transition in (3.5.3) from the coordinate r to the variable $\rho \equiv r + GM$ transforms (3.5.3) to the form that is adequate to (3.5.2):

$$ds^2 = \left(1 - \frac{2GM}{\rho}\right) dt^2 - \left(1 - \frac{2GM}{\rho}\right)^{-1} d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{3.5.2a}$$

They differ markedly in content, however, since in (3.5.2a) corresponding to the centre S of M is the value $\rho_S = +GM$, and in (3.5.2) $r_S = 0$. We have a similar procedure in passing in (3.5.2) from r to $\rho \equiv r - GM$ so that (3.5.2) will be reduced to an expression that is similar to (3.5.3) in form, but in essence equivalent to (3.5.3), since the transition changes the value $r_S = 0$ to $\rho_S = -GM$.

Using standard methods and confining ourselves in calculations to the first order in G, will give the time of propagation in one direction (at $\varphi_2 - \varphi_1 > \pi/2$,

when pericentre coincides with r_f) [57, 58] for the solution (3.5.2)

$$t = \sqrt{r_p^2 - r_f^2} + \sqrt{r_e^2 - r_f^2} + GM\left\{2\ln\frac{r_p + \sqrt{r_p^2 - r_f^2}}{r_e - \sqrt{r_e^2 - r_f^2}} + \left[\left(\frac{r_p - r_f}{r_p + r_f}\right)^{1/2} + \left(\frac{r_e - r_f}{r_e + r_f}\right)^{1/2}\right]\right\} \tag{3.5.4}$$

and for the solution (3.5.3)

$$t = \sqrt{r_p^2 - r_f^2} + \sqrt{r_e^2 - r_f^2} + GM\left\{2\ln\frac{r_p + \sqrt{r_p^2 - r_f^2}}{r_e - \sqrt{r_e^2 - r_f^2}} + 2\left[\left(\frac{r_p - r_f}{r_p + r_f}\right)^{1/2} + \left(\frac{r_e - r_f}{r_e + r_f}\right)^{1/2}\right]\right\}. \tag{3.5.5}$$

We pass in (3.5.4) and (3.5.5) from points $r_{e,p,f}$ to physically observable quantities. Using respectively the metric (3.5.2) and (3.5.3), we will, in the same first order in G, compute real physical distances (measurable) from the surface r_f to r_e and r_p:

$$l_{e,p} = \int_{r_f}^{r_{e,p}} dr\sqrt{-g_{rr}} = r_{e,p} - r_f + GM\ln(r_{e,p}/r_f). \tag{3.5.6}$$

We see that in the first order in G they will be the same for both metrics. We will also compute (in the same first order in G, of course) the relative frequency shift (measurable) in the gravitational field of source M

$$\left.\frac{\Delta\omega}{\omega}\right|_{e,p} \equiv \delta_{e,p} = GM\left(\frac{1}{r_f} - \frac{1}{r_{e,p}}\right). \tag{3.5.7}$$

The shift is again the same (for the accuracy chosen) for both metrics. Using (3.5.6) and (3.5.7), we can now express in (3.5.4) and (3.5.5) $r_{e,p,f}$ through the measurable quantities $l_{e,p}$ and $\delta_{e,p}$. Comparing then the times t of the propagation of a radiosignal from e to p, computed from (3.5.4) and (3.5.5), we see that they are expressly dissimilar, which attests to the ambiguity of GTR is predictions for t, the ambiguity manifests itself in this effect in first-order quantities in G.

For $r_f \ll r_{e,p}$, from (3.5.4) and (3.5.5), taking into account (to the accuracy chosen) the effect of the deflection of the signal by the gravitational field of a source, we obtain

$$t = R + 2GM\ln\frac{r_e + r_p + R}{r_e + r_p - R} - 2GM, \tag{3.5.4a}$$

$$t = R + 2GM\ln\frac{r_e + r_p + R}{r_e + r_p - R}, \tag{3.5.5a}$$

where

$$R = \sqrt{r_e^2 - r_\perp^2} + \sqrt{r_p^2 - r_\perp^2} \tag{3.5.8}$$

is the relative distance (along a straight line) between points e and p, and $r_\perp$ is the

coordinate of the point where the straight lines meet, which connect e and p, on the one hand, and S with the pericentre of the signal trajectory, on the other hand.

Analysis of other known gravitational effects given in [58-61] indicates that in the class of solutions $C(r) = [1 + (\lambda + 1) \times (GM/r)]^2$, where λ is a free parameter, the ambiguity of GTR predictions shows up in all effects without exception.

The main underlying ideas of RTG can be recapitulated as follows:

(I) In RTG the fundamental, basis space in the Minkowskian space x^μ (with the metric $\gamma^{\mu\nu}$). This property is inherent in all matter, whatever its nature; this is the property of universality of the laws of conservation of energy-momentum and angular momentum separately.

(II) The gravitational field in RTG is regarded as a real (with a zero rest mass) physical field in this space with all the features inherent in other physical fields; corresponding to it is the field symmetrical tensor $\Phi^{\mu\nu}$ of the second rank with representations corresponding to spin states 2 and 0.

(III) The Lagrangian density of other forms of matter (except for GF), by virtue of the universality of gravitational interactions and tensor character of GF, is constructed in RTG on the basis of convolutions with the effective tensor $g^{\mu\nu}$, defined by "addition" of the gravitational field $\Phi^{\mu\nu}$ to the metric tensor $\gamma^{\mu\nu}$ by the rule

$$\tilde{g}^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu} \equiv \sqrt{-\gamma}\, \gamma^{\mu\nu} + \sqrt{-\gamma}\, \Phi^{\mu\nu} \equiv \gamma^{\mu\nu} + \tilde{\Phi}^{\mu\nu}. \tag{3.5.9}$$

In the Lagrangian the derivatives of nongravitational physical fields are assumed to be covariant derivatives ∇_μ with respect to the effective metric $g^{\mu\nu}$.

This statement, which can conveniently be referred to as the "principle of geometrization", introduces into the theory, as a consequence of the universality of gravitational interactions and the tensor nature of GF, the secondary notion of the effective Riemannian space (with metric $g^{\mu\nu}$ defined in one map), which has apparently purely field origin. Secondary notions in the theory are still the Minkowskian space (with metric $\gamma^{\mu\nu}$) and the gravitational field $\Phi^{\mu\nu}$ in the space. The principle of geometrization in RTG is not adequate to the equivalence principle in GTR, since in RTG, like in other physical field theories, owing to the tensor (and not pseudo-tensor) nature of all physical quantities obtained using RTG the energy density of GF at a point cannot be made to vanish by any coordinate transformations, although the force action of GF at the point can be compensated for.

(IV) The density of the Lagrangian of the gravitational field is in RTG assumed to be a quadratic function of the first-order derivatives $D_\lambda \tilde{g}^{\mu\nu}$ that are covariant in the metric $\gamma^{\mu\nu}$ of the Minkowskian space.

Using statement (I)–(IV) we can construct the relativistic theory of gravitation in a unique manner.

The most direct way to construct the scalar density $\mathcal{L}_g(\gamma^{\mu\nu}, \tilde{g}^{\mu\nu}, D\lambda\tilde{g}^{\mu\nu})$ that meets item (IV), with items (I)–(III) taken into account, of the Lagrangian of the free GF in a Minkowskian space would be to represent it in the form of a general superposition of the various convolutions of form that are quadratic in the first-order derivatives $D\lambda\tilde{g}^{\mu\nu}$ with the tensors $\tilde{g}_{\alpha\beta}$ and $\tilde{\gamma}_{\alpha\beta}$ (see, e.g., [56, 62]*). We will here follow another procedure: we will seek the structure $\mathcal{L}_g$ using

*Using the gauge principle, the Lagrangian density $\mathcal{L}_g$ is thus determined uniquely.

from the very beginning the gauge principle [62], which requires that the Lagrangian density for the free GF, under transformations of the form

$$\delta_\epsilon \tilde{\Phi}^{\mu\nu} \equiv \delta_\epsilon \tilde{g}^{\mu\nu} = \tilde{g}^{\mu\lambda} D_\lambda \epsilon^\nu(x) + \tilde{g}^{\nu\lambda} D_\lambda \epsilon^\mu(x) - D_\lambda(\epsilon^\lambda \tilde{g}^{\mu\nu}) \qquad (3.5.10)$$

would change by no more than the divergence*.

$$\mathcal{L}_g \to \mathcal{L}_g + D_\nu Q^\nu(x).$$

In (3.5.10) $\epsilon^\nu(x)$ are infinitesimal parameters of the gauge transformation, and the operators δ_ϵ obeying

$$(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})\tilde{g}^{\mu\nu}(x) = \delta_{\epsilon_3}\tilde{g}^{\mu\nu}(x),$$

where

$$\epsilon_3^\nu = \epsilon_1^\mu D_\mu \epsilon_2^\nu - \epsilon_2^\mu D_\mu \epsilon_1^\nu,$$

form the Lie algebra. At the same time we will have to take into account the requirement of (II) that representations corresponding to spin values 1 and 0′ be excluded from among the states of the field $\Phi^{\mu\nu}$, which can be done by subjecting $\Phi^{\mu\nu}$ to the field equation

$$D_\mu \Phi^{\mu\nu} = \frac{1}{\sqrt{-\gamma}} D_\mu \tilde{g}^{\mu\nu} = 0. \qquad (3.5.11)$$

It should be stressed that equation (3.5.11) not only excludes from consideration nonphysical spin states of the gravitational field $\Phi^{\mu\nu}$, but also makes the metric $\gamma^{\mu\nu}$ of the Minkowski space unremovable from the theory**, thus enabling manifestations of GF. At the same time the field equations (3.5.11) narrow down the class of possible gauge transformations (3.5.10) to the manifold of the four-vector $\epsilon^\nu(x)$ that obey

$$g^{\alpha\beta} D_\alpha D_\beta \epsilon^\nu(x) = 0. \qquad (3.5.12)$$

We now address ourselves to constructing equations that, together with (3.5.11), form a system of main equations of GF. We will consider that the simplest densities that change under transformations (3.5.10) by a divergent quantity, are

$$\sqrt{-g} \to \sqrt{-g} - D_\nu(\epsilon^\nu \sqrt{-g})$$

and

$$\sqrt{-g}\,R \to \sqrt{-g}\,R - D_\nu(\epsilon^\nu \sqrt{-g}\,R),$$

where R is the scalar curvature of the effective Riemannian space-time given by

$$\sqrt{-g}\,R = \sqrt{-g}\,g^{\mu\nu}[(\Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}) + \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\sigma}] =$$
$$= \tilde{g}^{\mu\nu}(G^\lambda_{\mu\nu}G^\sigma_{\lambda\sigma} - G^\lambda_{\mu\sigma}G^\sigma_{\nu\lambda}) - D_\nu(\tilde{g}^{\mu\nu}G^\sigma_{\mu\sigma} - \tilde{g}^{\mu\sigma}G^\nu_{\mu\sigma}),$$

*The gauge transformation (3.5.10) of the field $\tilde{\Phi}^{\mu\nu}$ differs markedly from its coordinate $(x \to x + \xi)$ transformation

$$\delta_\epsilon \tilde{\Phi}^{\mu\nu} = \tilde{\Phi}^{\mu\lambda} D_\lambda \xi^\nu(x) + \tilde{\Phi}^{\nu\lambda} D_\lambda \xi^\mu(x) - D_\lambda(\xi^\lambda \tilde{\Phi}^{\mu\nu}).$$

**This suggests that equation (3.5.11) cannot have anything to do with the coordinate conditions.

where

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}),$$

$$G^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(D_\mu g_{\sigma\nu} + D_\nu g_{\sigma\mu} - D_\sigma g_{\mu\nu}).$$

We can also add the term $\sim\gamma_{\mu\nu}\tilde{g}^{\mu\nu}$, that, by (3.5.11), also satisfies the gauge principle. In the general case, we then can represent $\mathcal{L}_g$ in the form

$$\mathcal{L}_g = \lambda_1(\sqrt{-g}\,R + D_\nu Q^\nu(x)) + \lambda_2\sqrt{-g} + \lambda_3\gamma_\mu\tilde{g}^{\mu\nu} + \lambda_4\sqrt{-\gamma}. \tag{3.5.13}$$

Here the divergent term

$$D_\nu Q^\nu(x) = D_\nu(\tilde{g}^{\mu\nu}G^{\sigma}_{\mu\sigma} - \tilde{g}^{\mu\sigma}G^{\nu}_{\mu\sigma})$$

is added (using the gauge principle) in order to exclude from $\mathcal{L}_g$ terms with second derivatives that enter into $\sqrt{-g}\,R$, and the meaning of the remaining quantities will become clear later.

Lagrangian (3.5.13) gives the following expression for the tensor of energy-momentum of GF in the Minkowski space

$$t^{\mu\nu}_{(g)} \equiv -2\frac{\delta\mathcal{L}_g}{\delta\gamma_{\mu\nu}} = 2\sqrt{-\gamma}\left(\gamma^{\mu\alpha}\gamma^{\nu\beta} - \frac{1}{2}\gamma^{\mu\nu}\gamma^{\alpha\beta}\right)\frac{\delta\mathcal{L}_g}{\delta\tilde{g}^{\alpha\beta}}$$
$$+ \lambda_1 J^{\mu\nu} - 2\lambda_3\tilde{g}^{\mu\nu} - \lambda_4\tilde{\gamma}^{\mu\nu}, \tag{3.5.14}$$

where

$$J^{\mu\nu} \equiv D_\alpha D_\beta(\gamma^{\alpha\mu}\tilde{g}^{\beta\nu} + \gamma^{\alpha\nu}\tilde{g}^{\beta\mu} - \gamma^{\alpha\beta}\tilde{g}^{\mu\nu} - \gamma^{\mu\nu}\tilde{g}^{\alpha\beta}). \tag{3.5.15}$$

By the least-action principle, we will obtain from (3.5.14) two versions of the dynamic equation for GF that are different in form but identical in content

$$\frac{\delta\mathcal{L}_g}{\delta\tilde{g}^{\mu\nu}} = \lambda_1 R_{\mu\nu} + \frac{1}{2}\lambda_2 g_{\mu\nu} + \lambda_3\gamma_{\mu\nu} = 0, \tag{3.5.16}$$

where

$$R_{\mu\nu} = D_\lambda G^{\lambda}_{\mu\nu} - D_\mu G^{\lambda}_{\nu\lambda} + G^{\sigma}_{\mu\nu}G^{\lambda}_{\sigma\lambda} - G^{\sigma}_{\mu\lambda}G^{\lambda}_{\nu\sigma}, \tag{3.5.17}$$

and

$$\lambda_1 J^{\mu\nu} - 2\lambda_3\tilde{g}^{\mu\nu} - \lambda_4\tilde{\gamma}^{\mu\lambda} = t^{\mu\nu}_{(g)}. \tag{3.5.18}$$

In order that equation (3.5.16) may be obeyed identically and that $t^{\mu\nu}_{(g)} \equiv 0$ in the absence of a gravitational field, we must put $\lambda_4 = -2\lambda_3$, $\lambda_2 = -2\lambda_3$. Values λ_1 and λ_3 can readily be identified by writing (3.5.18), from (3.5.11) and (3.5.9),

$$\gamma^{\alpha\beta}D_\alpha D_\beta\tilde{\Phi}^{\mu\nu} + 2\frac{\lambda_3}{\lambda_1}\tilde{\Phi}^{\mu\nu} = -\frac{1}{\lambda_1}t^{\mu\nu}_{(g)}. \tag{3.5.19}$$

Equation (3.5.19) looks especially dramatic in Galilean coordinates

$$\Box\,\Phi^{\mu\nu} + 2\frac{\lambda_3}{\lambda_1}\Phi^{\mu\nu} = -\frac{1}{\lambda_1}t^{\mu\nu}_{(g)}. \tag{3.5.19a}$$

It is perhaps natural to attach to the factor $(+2\lambda_3/\lambda_1)$ the meaning of the squared rest mass of GF, i.e., to put $(+2\lambda_3/\lambda_1) \doteq m^2$, and to put $(-1/\lambda_1)$, by the principle of conformity, to be equal to 16π, i.e.,

$$\lambda_1 = -\frac{1}{16\pi}, \quad \lambda_2 = \lambda_4 = +\frac{m^2}{16\pi}, \quad \lambda_3 = \frac{-m^2}{32\pi}.$$

Thus, the Lagrangian of a free *GF* constructed on the basis of the gauge principle in the Minkowski space will in the general case have the form

$$\mathcal{L}_g = \frac{1}{16\pi}\tilde{g}^{\mu\nu}(G^\lambda_{\mu\nu}G^\sigma_{\lambda\sigma} - G^\lambda_{\mu\sigma}G^\sigma_{\nu\lambda}) - \frac{m^2}{16\pi}\left(\frac{1}{2}\gamma_{\mu\nu}\tilde{g}^{\mu\nu} - \sqrt{-g} - \sqrt{-\gamma}\right). \tag{3.5.20}$$

The corresponding dynamic equations of *GF*, which are complementary to (3.5.11), can be represented by two completely equivalent forms

$$R^{\mu\nu} - \frac{m^2}{2}(g^{\mu\nu} - g^{\mu\alpha}g^{\nu\beta}\gamma_{\alpha\beta}) = 0, \tag{3.5.21}$$

or

$$D_\alpha D_\beta(\gamma^{\alpha\beta}\tilde{g}^{\mu\nu} + \gamma^{\mu\nu}\tilde{g}^{\alpha\beta} - \gamma^{\alpha\mu}\tilde{g}^{\beta\nu} - \gamma^{\alpha\nu}\tilde{g}^{\beta\mu}) + \\ + m^2(\tilde{g}^{\mu\nu} - \tilde{\gamma}^{\mu\nu}) = 16\pi t^{\mu\nu}_{(g)}. \tag{3.5.22}$$

It follows that for a gravitational field with a rest mass the conservation laws of the energy-momentum $D_\mu t^{\mu\nu}_{(g)} = 0$ will only occur when conditions (3.5.11) are met. It is to be emphasized that equations (3.5.21) and (3.5.22) are not gauge-invariant, even if $\epsilon^\nu(x)$ obeys (3.5.12). This means that the introduction into the Lagrangian of a mass term removes degeneration * and uniquely defines the geometry of space-time, and also the density of the tensor of the energy-momentum of *GF*.

The complete system of equations for a free GF has the form

$$R^{\mu\nu} - \frac{m^2}{2}(g^{\mu\nu} - g^{\mu\alpha}g^{\nu\beta}\gamma_{\alpha\beta}) = 0, \qquad D_\mu\tilde{g}^{\mu\nu} = 0, \tag{3.5.23}$$

or, in another equivalent form,

$$\gamma^{\alpha\beta}D_\alpha D_\beta\tilde{\Phi}^{\mu\nu} + m^2\tilde{\Phi}^{\mu\nu} = 16\pi t^{\mu\nu}_{(g)}, \qquad D_\mu\tilde{\Phi}^{\mu\nu} = 0. \tag{3.5.24}$$

By virtue of (III) and (IV), in the presence of two forms of matter the total Lagrangian density is given by

$$\mathcal{L} = \mathcal{L}_M(\tilde{g}^{\mu\nu}, \Phi_a) + \mathcal{L}_g(\tilde{\gamma}^{\mu\nu}, \tilde{g}^{\mu\nu}, D_\lambda\tilde{g}^{\mu\nu}), \tag{3.5.25}$$

where Φ_a are matter fields (except for *GF*) and $\mathcal{L}_g$ is given by (3.5.20). This yields the following dynamic equations:

$$D_\alpha D_\beta(\gamma^{\alpha\beta}\tilde{g}^{\mu\nu} + \gamma^{\mu\nu}\tilde{g}^{\alpha\beta} - \gamma^{\alpha\mu}\tilde{g}^{\nu\beta} - \gamma^{\alpha\nu}\tilde{g}^{\beta\mu}) + m^2\tilde{\Phi}^{\mu\nu} = 16\pi t^{\mu\nu}. \tag{3.5.26}$$

Here $t^{\mu\nu}$ is the density of the symmetric tensor of the energy-momentum for all matter ($t^{\mu\nu} = t^{\mu\nu}_{(g)} + t^{\mu\nu}_{(M)}$) in the Minkowski space. Equations (3.5.26) can identi-

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* Without the mass term equation (3.5.22) is gauge-invariant.

cally be reduced to

$$R^{\mu\nu} - \frac{m^2}{2}(g^{\mu\nu} - g^{\mu\alpha}g^{\nu\beta}\gamma_{\alpha\beta}) = \frac{8\pi G}{\sqrt{-g}}\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right), \tag{3.5.27}$$

where $T^{\mu\nu} = -2(\delta \mathcal{L}_M/\delta g_{\mu\nu})$ is the density of the energy-momentum tensor of the nongravitational forms of matter in the effective Riemannian space. If we take into account the field equations (3.5.11), we will arrive at the system of main dynamic equations of RTG

$$\gamma^{\alpha\beta}D_\alpha D_\beta \tilde{g}^{\mu\nu} + m^2\tilde{\Phi}^{\mu\nu} \equiv \gamma^{\alpha\beta}D_\alpha D_\beta \tilde{\Phi}^{\mu\nu} + m^2\tilde{\Phi}^{\mu\nu} = 16\pi t^{\mu\nu}, \tag{3.5.28}$$

$$D_\mu \tilde{g}^{\mu\nu} \equiv D_\mu \tilde{\Phi}^{\mu\nu} = 0, \tag{3.5.29}$$

or equivalently,

$$R^{\mu\nu} - \frac{m^2}{2}(g^{\mu\nu} - g^{\mu\alpha}g^{\nu\beta}\gamma_{\alpha\beta}) = \frac{8\pi G}{\sqrt{-g}}\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right), \tag{3.5.28a}$$

$$D_\mu \tilde{g}^{\mu\nu} = 0. \tag{3.5.29a}$$

These equations are in principle different from those in GTR and they are equal in their significance. In (3.5.28), (3.5.29) or (3.5.28a), (3.5.29a) all the field variables are functions of the coordinates of the Minkowski space and the metric $\gamma^{\mu\nu}$ of the space enters into the equations in an unremovable manner. Given the boundary and initial conditions, the solution of the main system of RTG will have the property of uniqueness *, and so the physical quantities and predictions obtained using the system will be unique as well. By the field equations (3.5.29), we obtain directly from (3.5.28) the conservation law for the energy-momentum $D_\mu t^{\mu\nu} = 0$ that contains no ambiguities [63], since the metric $\gamma^{\mu\nu}$ of the Minkowski space enters organically into the RTG equations. We reiterate that this law only occurs provided the field equations (3.5.29) are obeyed. If we proceed from (3.5.28a), then, taking into account the relations

$$\nabla_\lambda \gamma_{\mu\nu} = -G^\sigma_{\lambda\mu}\gamma_{\sigma\nu} - G^\sigma_{\lambda\nu}\gamma_{\mu\sigma}, \quad D_\mu \tilde{g}^{\mu\nu} \equiv \sqrt{-g}\,(D_\mu g^{\mu\nu} + G^\lambda_{\mu\lambda}g^{\mu\nu}) = 0,$$

where ∇_λ is the covariant derivative with respect to the metric $g_{\mu\nu}$ of the effective Riemannian space, we will come to another form of the conservation law

$$\nabla_\mu T^{\mu\nu} = 0.$$

We then assume formally the rest mass of GF to be zero. Then

$$\mathcal{L}_g = \frac{1}{16\pi}\tilde{g}^{\mu\nu}(G^\lambda_{\mu\nu}G^\sigma_{\lambda\sigma} - G^\lambda_{\mu\sigma}G^\sigma_{\nu\lambda}), \tag{3.5.20a}$$

and the dynamic equations of the free *GF* will become

$$D_\alpha D_\beta(\gamma^{\alpha\beta}\tilde{g}^{\mu\nu} + \gamma^{\mu\nu}\tilde{g}^{\alpha\beta} - \gamma^{\alpha\mu}\tilde{g}^{\nu\beta} - \gamma^{\alpha\nu}\tilde{g}^{\beta\mu}) = 16\pi t^{\mu\nu}_{(g)}. \tag{3.5.22a}$$

This equation is invariant under permissible gauge transformations. At the same time, the tensor $t^{\mu\nu}_{(g)}$ of the field will not be gauge invariant, but because its change $\delta_\epsilon t^{\mu\nu}_{(g)}$ under the transformation (3.5.10), as is easily seen from (3.5.22a), leads

*From the equation of state of matter, the system (3.5.28), (3.5.29) or (3.5.28a), (3.5.29a) becomes a closed system of equations that define the dynamics of the field and matter alike.

to the divergence of the skew-symmetric tensor of rank three

$$\delta_\epsilon t^{\mu\nu}_{(g)} = -\frac{1}{4\pi} D_\lambda D_\sigma \delta_\epsilon \Pi^{[\mu\sigma][\nu\mu]},$$

where

$$\Pi^{[\mu\sigma][\nu\lambda]} = \frac{1}{4}(\gamma^{\lambda\mu}\tilde{g}^{\nu\sigma} + \gamma^{\sigma\nu}\tilde{g}^{\mu\lambda} - \gamma^{\lambda\sigma}\tilde{g}^{\mu\nu} - \gamma^{\mu\nu}\tilde{g}^{\lambda\sigma}),$$

the gauge arbitrariness $t^{\mu\nu}_{(g)}$ will not effect the integral physical characteristics we seek [56]. Other quantities that will not be gauge invariant are the interval of the effective Riemannian space-time and its respective geometric characteristics [62].

In the presence of other forms of matter the total Lagrangian density will be determined by the expression (3.5.25) where $\mathcal{L}_g$ is from (3.5.20a). The corresponding dynamic equations become

$$D_\alpha D_\beta(\gamma^{\alpha\beta}\tilde{g}^{\mu\nu} + \gamma^{\mu\nu}\tilde{g}^{\alpha\beta} - \gamma^{\alpha\mu}\tilde{g}^{\nu\beta} - \gamma^{\alpha\nu}\tilde{g}^{\beta\mu}) = 16\pi t^{\mu\nu}.$$

Owing to the presence of matter these equations will no longer be gauge invariant, and so the theory will no longer be gauge arbitrary [62].

In combination with (3.5.11) the system of main equations of RTG will have the form

$$\gamma^{\alpha\beta}D_\alpha D_\beta \tilde{g}^{\mu\nu} \equiv \gamma^{\alpha\beta}D_\alpha D_\beta \tilde{\Phi}^{\mu\nu} = 16\pi t^{\mu\nu}, \tag{3.5.30}$$

$$D_\mu \tilde{g}^{\mu\nu} \equiv D_\mu \tilde{\Phi}^{\mu\nu} = 0, \tag{3.5.31}$$

or equivalently,

$$\sqrt{-g}\,R^{\mu\nu} = 8\pi G\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right), \tag{3.5.30a}$$

$$D_\mu \tilde{g}^{\mu\nu} = 0. \tag{3.5.31a}$$

Although formally equations (3.5.30a) coincide with the Hilbert–Einstein equations, their meaning is different, since the field variables in them do not depend on the coordinates of the Minkowski space, and the joint system contains the metric of the Minkowski space in an unremovable manner.

That the complete system of equations of RTG (including the equations of matter and of GF) organically includes, in addition to the field variables of matter and the metric tensor of the effective Riemannian space, the metric tensor of the Minkowski space, is a circumstance of principle. It enables RTG to consider all physical fields, including the gravitational one, in a unified Minkowski space. In GTR it is impossible, because its equations do not contain the metric tensor of the Minkowski space.

The field equation (3.5.31) introduced into the theory thus makes the metric of the Minkowski space unremovable from the theory, which is reflected in the description of all physical phenomena and leads to consequences that are qualitatively different from GTR.

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